SMITHSONIAN CONTRIBUTIONS TO KNOWLEDGE.
199

AN INVESTIGATION OF THE ORBIT OF NEPTUNE,

WITH GENERAL TABLES OF ITS MOTION.

BY

SIMON NEWCOMB,

PROFESSOR OF MATHEMATICS, UNITED STATES NAVY.

[ACCEPTED FOR PUBLICATION, MAY, 1865.]

COMMISSION

TO WHICH THIS PAPER HAS BEEN REFERRED.

Admiral C. H. DAVIS, U. S. N.
Prof. STEPHEN ALEXANDER.

JOSEPH HENRY,
Secretary S. I.

COLLINS, PRINTER,
PHILADELPHIA.

TABLE OF CONTENTS.

CHAPTER I.

INTRODUCTION.

CHAPTER II.

PROVISIONAL THEORY OF NEPTUNE.

CHAPTER III.

DISCUSSIONS OF OBSERVATIONS OF NEPTUNE.

CHAPTER IV.

RESULTS OF THE COMPARISON OF THE THEORETICAL WITH THE OBSERVED POSITIONS OF NEPTUNE.

CHAPTER V.

TABLES OF NEPTUNE.

ON THE ORBIT OF NEPTUNE.

CHAPTER I.

INTRODUCTION.

The errors of the published ephemerides of Neptune are now increasing very rapidly. In 1863, Walker's ephemeris was in error by 33″, and Kowalski's by 22″. Both ephemerides may be 5′ in error before the end of the present century. The orbit of this planet is, therefore, more uncertain than that of any other of the larger members of our system. The uncertainty arises from the insufficiency of the data at the command of those astronomers who have hitherto investigated the motions of this planet. These motions are so slow that it is impossible to determine the elements of the orbit with accuracy from observations extending through only a few years. In Walker's investigations the errors of observation are multiplied more than a hundred times in the elements deduced from them, on account of the smallness of the arc through which the planet had moved.

The time has now come when the orbit can be determined with some approach to accuracy. The planet has moved through an arc of nearly 40° since its discovery, and the errors of observation will be multiplied only ten or twelve times in the errors of the elements. In commencing the work of a revision of the theory of Neptune, it will be well to glance at the past and present state of our knowledge on this subject.

Approximate elements of this planet, neglecting the effect of perturbations, were computed by several astronomers within a year or two after its discovery. But the work of preparing a theory which should include the perturbations produced by all the other planets seems to have been left entirely in the hands of Professor Peirce and Mr. Sears C. Walker.

§ 2. All the first approximations to the elements showed that the mean motion was very nearly half that of Uranus. It was, therefore, for some time doubtful whether the mutual action of the two planets might not be such as to render the period of Neptune exactly double that of Uranus, and thus present us, on a much grander scale, with a phenomenon similar to that exhibited by the satellites of Jupiter. Professor Peirce's first perturbations of Neptune were computed on this hypothesis, and published in the Monthly Notices of the Royal Astronomical Society, Vol. VIII, p. 40. The eccentricity of Neptune was neglected, but that of the disturbing planets was included in the perturbations.

With these perturbations, the ancient observations of Lalande, and the vast number of modern observations made in nearly every active observatory in the world during 1846 and 1847, Mr. Walker computed his "Elliptic Elements I." of

Neptune. The longitude of perihelion referred to the mean Equinox of Jan. 1, 1847, eccentricity, and mean daily motion were as follows:

$$\pi = 48° \, 21' \, 2''.93$$
$$e = .00857741.$$
$$n = 21''.55448.$$

This mean motion rendered it certain that the supposed relation between the mean motions of the planets Uranus and Neptune had no foundation in fact. Professor Peirce thereupon revised his theory, and published the new perturbations in the Proceedings of the American Academy, Vol. I, p. 286.

The near approach to commensurability of the mean motions renders the general theory of the mutual action of Uranus and Neptune extremely complex. Twice the mean motion of the latter exceeds that of the former by only 320″ according to Walker, or 304″ according to my first revision of his elements. The terms in the perturbations which contain this very small quantity as a divisor will, therefore, be very large. Considered as perturbations of the elements, their period will be more than 4000 years. We have an analogous instance in the 900 year equation of Jupiter and Saturn. But in the latter case the perturbations of the mean motion are of the *third* order with respect to the eccentricities and inclinations, while in Uranus and Neptune they are of the first order. From this circumstance it happens that, notwithstanding the smaller masses of the disturbing planets, the perturbation of the mean motion is as great in the case of the planets in question as in that of Jupiter and Saturn, and that of the other elements enormously greater. In fact, the perihelion of Neptune oscillates through a space of *eight degrees* in consequence of the terms in question. Such a perturbation as this, four degrees on each side of the mean, is, I think, found nowhere else in our system. Moreover, a change of 1″ in the mean motion of the planet will produce a change of nearly 2′ in the coefficient of this perturbation. Any attempt to determine its magnitude with accuracy will, therefore, be hopeless.

But the difficulties connected with these terms can be avoided in the case of a theory which is designed to be exact for a period of only a few centuries. Notwithstanding the great magnitude of the general integrals of the perturbations, if we take these integrals between limits not exceeding a couple of centuries, we shall find them so small as not to involve serious difficulty. Their effect on the co-ordinates can then be developed in powers of the time, and the values thus obtained will not be subject to any uncertainty of moment. This is substantially the course adopted by Professor Peirce. He says of the terms in question:

"These coefficients will vary very sensibly by a change in the value of the mean motion of Neptune, arising from a more accurate determination of its orbit. But the principal effect of these terms can for a limited period, such as a century, for instance, be included in the ordinary forms of elliptic motion, and the residual portion will assume a secular form which is no more liable to change from a new correction of the mean motion of Neptune than the other small coefficients of the equations of perturbations."

Accordingly, subducting from the terms in question a series of expressions

which would result from arbitrary changes in the elliptic orbit, there is left a small residual, mostly developed in powers of the time, and only amounting to a few seconds in a century, which alone is retained.

With the new perturbations, and revised normal places of Neptune, Mr. Walker obtained the following final set of elements, which he denominated Elliptic Elements II.:

$$
\begin{aligned}
\pi &= 47^\circ\ 12'\ \ 6''.50 \\
\Omega &= 130\ \ \ 4\ \ 20\ .81 \\
\varepsilon &= 328\ \ 32\ \ 44\ .20 \\
i &= 1\ \ 46\ \ 58\ .97. \\
e &= .00871946. \\
\mu &= 21''.55448.
\end{aligned}
$$

Epoch, Jan.1, 1847.

From these elements and perturbations we have a continuous ephemeris of Neptune since the time of its optical discovery. From 1846 till 1851 inclusive, this ephemeris is found in the Appendix to Vol. II of the Smithsonian Contributions to Knowledge; for 1852, in Vol. III of the same series, and also in the Astronomical Journal; and for subsequent years, in the American Ephemeris and Nautical Almanac.

All the modern observations on which these elements were founded were made in the years 1846–47, while the planet was moving over an arc of only two and a half degrees. Considering that the complete determination of the elements requires, effectively, four observed longitudes, all in different parts of the orbit, and that three of these positions are included in a space of less than three degrees, it must be admitted that an accurate determination of the elements was, under the circumstances, impossible, owing to the imperfections of the observations. As already remarked, the errors of observation would be multiplied several hundred times in the elements. Hence, with the best possible observations, the elements would be uncertain by one or more minutes. But the observations themselves were mainly differential ones; and it is very doubtful whether the positions of the stars of comparison were as well determined as the position of the planet itself could be determined by a series of good meridian observations.

§ 3. The theory of Neptune was next taken up by Professor Kowalski, of the University of Kasan. His work was published under the title of "*Recherches sur les mouvements de Neptune, suivées des tables de cette planète,* Kasan, 1855." The long-period perturbations of the elements are here developed, in their general form, as perturbations of the co-ordinates. There are, therefore, a much larger number of terms having large coefficients in this theory than in that of Professor Peirce.

Owing to this change in the form of the perturbations, the two theories cannot be directly compared. But the ephemerides resulting from each theory can be compared directly with observation, and corrections of the elements thence obtained. It is thus found that the elements in question require, approximately, the following corrections in order that the ephemerides may agree with observations to 1863:

	Theory of Walker.	Theory of Kowalski.
$\delta\pi$	$-4^\circ\ 11'\ ''$	$-4^\circ\ 12'\ ''$
δe	$-0\quad 0\ 52$	$-0\quad 0\ 51$
$\delta\varepsilon$	$-0\quad 3\ \ 6$	$-\quad 2\ 53$
δn	$-\quad 8.4$	$-\quad 8.5$

Thus, it seems that the theory of Kowalski is, on the whole, no nearer the truth than that of Walker, although it was founded on observations up to 1853, when the planet had moved through an arc of sixteen degrees since its optical discovery.* The cause of this failure to derive a more accurate result is an accidental mistake in the computation of the perturbations of the radius vector by Jupiter, as I have more fully pointed out in the Monthly Notices of the Royal Astronomical Society for December, 1864.

§ 4. The form which Professor Kowalski finds his equations of condition to assume is illustrative of an interesting and important principle of the method of least squares. By the comparison of his provisional theory with observations, forty-four equations of condition are obtained for the corrections of the four elements π, e, ε, and n. It is then inquired whether it is possible to determine the orbit of Neptune from the modern observations alone, omitting that of Lalande, the planet having moved through an arc of sixteen degrees. Treating the equations derived from the modern observations alone by the method of least squares, four normal equations are obtained. Two of these equations are, omitting the terms involving the correction of the mass of Uranus, which we do not need,

$$-10.4994\,\delta n - 21.2661\,\delta\varepsilon + 13.0088\,e\delta\pi + 40.2211\,\delta e = -324''.65,$$
$$26.9661\,\delta n - 73.2702\,\delta\varepsilon + 40.2211\,e\delta\pi + 139.9967\,\delta e = -886\,.63,$$

and the other two can be transformed into the following:

$$-10.4994\,\delta n - 21.2661\,\delta\varepsilon + 13.0073\,e\delta\pi + 40.2219\,\delta e = -324.50,$$
$$-26.9661\,\delta n - 73.2702\,\delta\varepsilon + 40.2219\,e\delta\pi + 140.0009\,\delta e = -886.77.$$

It will be seen that the last two equations are very nearly identical with the first two. Hence it is concluded that the modern observations alone give only two independent relations between the four unknown quantities sought, and do not suffice, therefore, to determine the elements of Neptune.

Now, the identity in question does not prove that the modern observations are insufficient to determine the elements, because *it is the necessary result of the mode of treating equations of the kind in question by the method of least squares.* This can be most easily shown by a theorem in determinants. By the elementary principle of determinants, if we have a number of linear equations between the same number of unknown quantities, of the form

* The differences of the two values of $\delta\pi$ and δe, which are so small, do not correctly represent the absolute differences of the two theories, owing to the great difference of longitude of perihelion in the two theories proceeding from the different forms given to the perturbations. The real difference Kowalski—Walker is given by the equations

$$\delta.e \sin\pi = +\ 1'',$$
$$\delta.e \cos\pi = -13.$$

$$
\begin{aligned}
a_1 x + b_1 y + c_1 z + \text{etc.} \ldots \ldots &= n_1, \\
a_2 x + b_2 y + c_2 z + \text{etc.} \ldots \ldots &= n_2, \\
\text{etc.} \quad \text{etc.} \quad \text{etc.} \quad \text{etc.} \qquad & \quad \text{etc.};
\end{aligned}
$$

each unknown quantity is given in the form

$$x = \frac{A_1}{R} n_1 + \frac{A_2}{R} n_2 + \frac{A_3}{R} n_3 + \text{etc.},$$

in which R represents the determinant formed from all the coefficients a, b, etc. in the given equations, and A_1, A_2, etc. the partial determinants, obtained by omitting column a, row 1, column a, row 2, etc.

If, now, the number of equations is greater than that of the unknown quantities, and they are solved by the method of least squares, the form of the solution will be the same as the above, except that for R will be substituted the sum of the squares of all the determinants R, formed by solving separately every combination of such number of the given equations as is equal to the number of unknown quantities, and for A_1, A_2, etc., certain powers and products of the partial determinants which enter into the separate solutions. Hence, if these determinants are very small, the corresponding determinants in the solution by least squares will be very small quantities of the second order. But the determinants will all be very small if the equations are nearly equivalent to a number less than that of the unknown quantities; that is, if they can be put into the form

$$
\begin{aligned}
X &= n_1, \\
Y &= n_2, \\
Z &= n_3, \\
\alpha X + \beta Y + \gamma Z + \text{etc.} + \rho &= n_4, \\
\alpha' X + \beta' Y + \gamma' Z + \text{etc.} + \rho' &= n_5, \\
\text{etc.} \quad \text{etc.} \quad \text{etc.} \quad \text{etc.} \quad \text{etc.} \quad & \text{etc.};
\end{aligned}
$$

the quantities X, Y, Z, etc. being less in number than the unknown quantities, and ρ being a very small linear function of the unknown quantities. If the ρ's vanish, all the determinants will vanish with it; whence, if they are very small, the determinants will be very small likewise. Calling a system of equations identical when they really give fewer independent relations than there are unknown quantities, the theorem sought may be expressed as follows:

If a system of equations differ from identity by a very small quantity, the normal equations derived from them will be identical to small quantities of the second order.

Hence, if such a system of equations is to be solved by least squares, it will be necessary to carry the solution to nearly twice as many decimals as are necessary in the original coefficients. Thus, in the case under consideration, as Professor Kowalski considered it necessary to retain four places of decimals in the coefficients of the unknown quantities, it would have been necessary to include at least six or seven decimals in the normal equations, instead of only four.

But the necessity for so long a numerical calculation can be avoided by a suitable transformation of the equations of condition. If the equations are identical, they really give certain linear functions of the unknown quantities less in number

than the unknown quantities. We may then substitute these linear functions themselves in place of an equal number of the unknown quantities. If the equations are not absolutely identical, the coefficients of the other unknown quantities will not entirely vanish by the substitution, and thus we shall still have the whole number of unknown quantities, only the coefficients of certain of them will be very small. The solution by least squares can then be performed without trouble, because the extra decimals will be necessary only in multiplying by the very small coefficients, when they can be introduced with ease. Afterward the values of the original unknown quantities can be deduced from those of the linear functions, and the unknown quantities which have been retained.

Suppose, for example, that the equations of condition are

$$\begin{array}{c}
a_1x + b_1y + c_1z = n_1 \\
a_2x + b_2y + c_2z = n_2 \\
a_3x + b_3y + c_3z = n_3 \\
a_4x + b_4y + c_4z = n_4 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \\
a_nx + b_ny + c_nz = n_n
\end{array}$$

A simple inspection, or, at least, an attempt to solve three of the most diverse of the equations, will show if the given n equations are really equivalent to only one or two. Then we should put

$$\begin{array}{l}
X = \alpha x + \beta y + \gamma z \\
Y = \alpha' x + \beta' y + \gamma' z
\end{array}$$

the coefficients α, β, γ, being entirely arbitrary, and so taken that when X and Y were substituted for x and y the coefficients of z should be as small as possible. It would conduce to simplicity if α and β', or α' and β, could each be made zero, which could always be done.

If we attempt to correct the elements of a planet's orbit by observations extending over only a few degrees, the equations of condition will necessarily be of the kind referred to. Hence a transformation of this kind will be advisable. An example will be given in the correction of the orbit of Neptune from observation.

§ 5. Ten years have elapsed since the publication of Kowalski's theory, and no general revision of the orbit has been published by any astronomer, so far as the writer is aware. The observations which have accumulated in the mean time would seem sufficient to fix the elements exactly enough to give the place of the planet within 5″ during the remainder of the present century. It is, therefore, proposed,

1. To determine the elements of the orbit of Neptune with as much exactness as a series of observations extending through an arc of forty degrees will admit of.
2. To inquire whether the mass of Uranus can be concluded from the motions of Neptune.

3. To inquire whether those motions indicate the action of an extra-Neptunian planet, or throw any light on the question of the existence of such a planet.

4. To construct general tables and formulæ by which the theoretical place of Neptune may be found at any time, and, more particularly, at any time between the years 1600 and 2000.

In giving the steps of an investigation like this, the true end should be to furnish the means whereby every step can be corrected, or verified if already correct, and to start only from admitted data. Sometimes a result will necessarily depend, to a certain extent, on an act of judgment, as in assigning relative values to different determinations of the same element. In this case data should be given for a revision of the judgment, as far as this may be thought desirable.

Such, with very few exceptions, is the rule adhered to in the present paper. The data are the published volumes of astronomical observations, and the fundamental formulæ of celestial mechanics. The steps will nearly or quite always be so short that any one may be verified from the preceding one without much labor.

The author is indebted to the courtesy of the Astronomer Royal, of the late Captain James M. Gilliss, and of Professor G. W. Hough, for the observations made at Greenwich, Washington, and Albany in the years 1863 and 1864, which have added greatly to the reliableness of the results of his investigation.

WASHINGTON, April, 1865.

CHAPTER II.

PROVISIONAL THEORY OF NEPTUNE.

§ 6. All the perturbations have been computed by formulæ founded on the method of La Grange; the development of the perturbative function in series, and the variation of arbitrary constants.

The following notation is used:

l = mean longitude.

λ = mean longitude, counted from ascending node of inner planet on outer one.

ϕ = inclination of orbit to the ecliptic.

γ = mutual inclination of two orbits to each other.

α = ratio of the mean distances.

$u = \sin \frac{1}{2} \gamma$.

f = mean anomaly.

ω = distance of the perihelion from the ascending node of the inner planet on the outer one.

For the other elements the almost universal notation of astronomers is adopted. The elements which pertain to the outer planet (Neptune) are distinguished by an accent.

The potential of the disturbing force exerted by one planet upon another, usually called the perturbative function, may be developed into an infinite series of terms, each of which shall be of the form

$$m \frac{h}{a'} \cos (i'\lambda' + i\lambda + j'\omega' + j\omega)$$

in which i, i', j, and j' are numerical coefficients. h is a function of the ratio of the mean distances, the eccentricities, and the mutual inclination of the orbits.

Then, by the theory of the variation of arbitrary constants, any term of the perturbative function in the action of an inner on an outer planet will cause the following differential variations of the four elements which determine the form of the orbit, and the position of the planet in it. Putting

$$i'\lambda' + i\lambda + j'\omega' + j\omega = N,$$
$$e = \sin \psi,\ g = \cos \psi \tan \tfrac{1}{2} \psi;$$

we have

$$\begin{aligned}
\frac{da'}{dt} &= -2\, mi'ha'n' \sin N, \\
\frac{d\varepsilon'}{dt} &= mn' \left\{ g' \frac{dh}{de'} + 2h + 2\alpha \frac{dh}{da} \right\} \cos N, \qquad (1) \\
\frac{de'}{dt} &= mn'h \left\{ j' \cot \psi' + i'g' \right\} \sin N, \\
\frac{d\pi'}{dt} &= mn' \cot \psi' \frac{dh}{de'} \cos N.
\end{aligned}$$

From the first equation and the relation between the mean distance and mean motion, we obtain

$$\frac{dn'}{dt} = 3\, m'n'^2\, i'h \sin N.$$

These equations are entirely rigorous, provided that we regard the elements in the second member as variable. But they can be integrated only by successive approximations. In a first approximation the elements are regarded as constant Equations similar to (1) for the elements of all the planets whose action is taken into account being integrated in this way, the resulting values may be substituted in the second members of (1), and a new integration be performed.

In the case of Neptune, however, the variations of the elements are so slow that a single approximation will be amply sufficient for a period of several centuries, provided that we adopt suitable values of the elements in the second members; that is, if we add such constants to the integrals that the latter shall be very small for the present time. Putting $\nu = \dfrac{n'}{i'n' + in'}$,

we shall have, on the supposition that the elements as they enter into the second member are constant,

$$\log a' = m\nu A \cos N + a'_0,\ e' = m\nu E \cos N + e'_0, \qquad (2)$$
$$l' = m\nu L \sin N + n'_0 t + \varepsilon'_0,\ \pi' = m\nu W \sin N + \pi'_0,$$

A, L, E, and W being given by the equations

$$\begin{aligned} A &= 2\, i'h, \\ L &= -3\, i'\nu h + 2\, h + 2\, \alpha \frac{dh}{d\alpha} + g \frac{dh}{de'}, \\ E &= -h\, (j' \cot \psi' + i'g'), \qquad (3) \\ W &= \cot \psi' \frac{dh}{de'}. \end{aligned}$$

a'_0, n'_0 ε', e'_0, and π'_0 are arbitrary constants, dependent on the position and velocity of the planet at a given epoch. a_0 and n_0 are, however, dependent on each other.

For the perturbations of the true longitude in orbit, and the logarithm of the radius vector, we shall have, omitting accents,

$$\begin{aligned} \delta v = \delta l & \\ + m\nu \{ eL - eW - E \} \sin (N - f) & \quad - \tfrac{1}{8} e^2 m\nu \{ eL - eW - 3E \} \sin (N - f) \\ + m\nu \{ eL - eW - E \} \sin (N + f) & \quad - \tfrac{1}{8} e^2 m\nu \{ eL - eW + 3E \} \sin (N + f) \\ + \tfrac{5}{4} em\nu \{ eL - eW - E \} \sin (N - 2f) & \quad - \text{etc.} \\ + \tfrac{5}{4} em\nu \{ eL - eW + E \} \sin (N + 2f) & \qquad (4) \\ + \tfrac{13}{8} e^2 m\nu \{ eL - eW - E \} \sin (N - 3f) & \\ + \tfrac{13}{8} e^2 m\nu \{ eL - eW + E \} \sin (N + 3f) & \\ + \tfrac{103}{48} e^3 m\nu \{ eL - eW - E \} \sin (N - 4f) & \\ + \quad \text{etc.} & \end{aligned}$$

$$
\begin{aligned}
\delta \log r = \delta \log a & \\
+ m\nu \{ 2\, i'h + \tfrac{1}{2} eE - \tfrac{1}{8} e^3 E \} \cos N & \quad - \tfrac{3}{16} e^2 m\nu \{ eL - eW - 3E \} \cos (N - f) \\
+ \tfrac{1}{2} m\nu \{ eL - eW - E \} \cos (N - f) & \quad + \tfrac{3}{16} e^2 m\nu \{ eL - eW + 3E \} \cos (N + f) \\
- \tfrac{1}{2} m\nu \{ eL - eW + E \} \cos (N + f) & \quad - \text{ etc.} \\
+ \tfrac{3}{4} em\nu \{ eL - eW - E \} \cos (N - 2f) & \qquad\qquad (5) \\
- \tfrac{3}{4} em\nu \{ eL - eW + E \} \cos (N + 2f) & \\
+ \tfrac{17}{16} e^2 m\nu \{ eL - eW - E \} \cos (N - 3f) & \\
- \text{ etc.} &
\end{aligned}
$$

By these formulæ all the perturbations of the longitude and radius vector have been computed, except that the computation was so conducted as to reject all terms above a certain order with respect to the eccentricities. The sum of all the factors (functions of the ratio of the mean distance) of any power of the eccentricity in any coefficient in the perturbations of the co-ordinates will generally be much smaller than each individual factor, as we shall presently show. If, for example, we have

$$\delta v = e^2 (f + f' + f'') \sin N$$

the sum $f + f' + f''$ will, in general, nearly destroy itself, being much smaller than the individual components, f, f', and f''. Hence, if the computation is arranged so as to include any one of the f's, it should include all. This end may be attained by omitting from h, its differential coefficients, and $h \cot \psi$, all terms of a higher order with respect to the eccentricities than the assigned limit. Thus, h being of the form

$$h = e'^s (x_1 + e'^2 x_2 + e'^4 x_4 + \ldots \}$$

if we limit ourselves to the power $s + 1$, we should put

$$
\begin{aligned}
& h = e'^s x_1; \ \alpha \frac{dh}{d\alpha} = e'^s \alpha \frac{dx_1}{d\alpha}; \\
& \frac{dh}{de'} = s e'^{s-1} x_1 + (s + 2)\, e'^{s+1} x_2 \\
& sh \cot \psi = s e'^{s-1} x_1 + s e'^{s+1} (-\tfrac{1}{2} x_1 + x_2).
\end{aligned}
$$

§ 7. *Perturbations of latitude.*

The equations which determine the change in the plane of a planet's orbit are

$$
\begin{aligned}
\frac{d\theta'}{dt} &= \frac{a'n'}{\sin \phi' \cos \psi'} \cdot \frac{dR}{d\phi'} \\
\frac{d\phi'}{dt} &= -\frac{a'n'}{\sin \phi' \cos \psi'} \cdot \frac{dR}{d\theta'}
\end{aligned}
\qquad (6)
$$

R being a function of $\lambda, \lambda', \omega, \omega'$, and γ, each of which depends on the position of the plane of the orbit, we have

$$
\begin{aligned}
\frac{dR}{d\phi'} &= \frac{dR}{d\lambda} \frac{d\lambda}{d\phi'} + \frac{dR}{d\omega} \frac{d\omega}{d\phi'} + \frac{dR}{d\lambda'} \frac{d\lambda'}{d\phi'} + \frac{dR}{d\omega'} \frac{d\omega'}{d\phi'} + \frac{dR}{d\gamma} \frac{d\gamma}{d\phi'} \\
\frac{dR}{d\theta'} &= \frac{dR}{d\lambda} \frac{d\lambda}{d\theta'} + \frac{dR}{d\omega} \frac{d\omega}{d\theta'} + \frac{dR}{d\lambda'} \frac{d\lambda'}{d\theta'} + \frac{dR}{d\omega'} \frac{d\omega'}{d\theta'} + \frac{dR}{d\gamma} \frac{d\gamma}{d\theta'}
\end{aligned}
$$

The values of the second of each pair of differential coefficients can easily be determined geometrically. λ, ω, λ', etc., it will be remembered, represent the distance of certain points on each orbit from the ascending node of the disturbing planet on the disturbed one: the infinitesimal changes in those quantities, produced by infinitesimal changes in the position of the plane of either orbit, will be due entirely to the changes in the position of that node. Let us put

$\varkappa'$ = distance of common node from ascending node of disturbed planet on the ecliptic.

$\varkappa$ = same quantity for disturbing planet.

By drawing the diagram, it will readily be seen that by a change in ϕ' the common node will be moved forward on the disturbed planet by the amount

$$+\sin\varkappa'\cot\gamma d\phi',$$

and on the disturbing planet by the amount

$$+\sin\varkappa'\operatorname{cosec}\gamma d\phi',$$

while γ will be varied by the amount

$$-\cos\varkappa' d\phi'.$$

In like manner, by a change in θ', the corresponding changes will be

$$-\cos\varkappa'\sin\phi'\cot\gamma d\theta',$$
$$-\cos\varkappa'\sin\phi'\operatorname{cosec}\gamma d\theta',$$
$$-\sin\varkappa'\sin\phi' d\theta'.$$

We therefore have

$$\frac{d\lambda'}{d\phi'}=\frac{d\omega'}{d\phi'}=-\sin\varkappa'\cot\gamma,$$
$$\frac{d\lambda}{d\phi'}=\frac{d\omega}{d\phi'}=-\sin\varkappa'\operatorname{cosec}\gamma,$$
$$\frac{1}{\sin\phi'}\frac{d\lambda'}{d\theta'}=\frac{1}{\sin\phi'}\frac{d\omega'}{d\theta'}=\cos\varkappa'\cot\gamma,$$
$$\frac{1}{\sin\phi'}\frac{d\lambda}{d\theta'}=\frac{1}{\sin\phi'}\frac{d\omega}{d\theta'}=\cos\varkappa'\operatorname{cosec}\gamma,$$
$$\frac{d\gamma}{d\phi'}=-\cos\varkappa';\ \frac{1}{\sin\phi'}\frac{d\gamma}{d\theta'}=-\sin\varkappa'.$$

Also, by the differentiation of the representative term of R,

$$\frac{dR}{d\lambda'}=-\frac{mi'h}{a'}\sin N,\qquad \frac{dR}{d\omega'}=-\frac{mj'h}{a'}\sin N,$$
$$\frac{dR}{d\lambda}=-\frac{mih}{a'}\sin N,\qquad \frac{dR}{d\omega}=-\frac{mjh}{a'}\sin N,$$
$$\frac{dR}{d\gamma}=\frac{dR}{du}\frac{du}{d\gamma}=\frac{1}{2}\frac{m}{a'}\frac{dh}{du}\cos\tfrac{1}{2}\gamma\cos N.$$

Substituting these expressions for the differential coefficients in the values of

$$\frac{dR}{d\phi'} \text{ and } \frac{dR}{d\theta'}, \text{ we have}$$

$$\frac{dR}{d\phi'} = \frac{m}{a'} h \sin x' \sin N \{ (i'+j') \cot \gamma + (i+j) \operatorname{cosec} \gamma \} - \tfrac{1}{2} \frac{m}{a'} \frac{dh}{du} \cos \tfrac{1}{2}\gamma \cos x' \cos N.$$

$$\frac{1}{\sin \phi} \frac{dR}{d\theta} = -\frac{m}{a'} h \cos x' \sin N \{ (i'+j') \cot \gamma + (i+j) \operatorname{cosec} \gamma \} - \tfrac{1}{2} \frac{m}{a'} \frac{dh}{du} \cos \tfrac{1}{2}\gamma \sin x' \cos N.$$

Let us now put

$$i'+j'+i+j = -\iota.$$

It may be remarked that ι will then be the coefficient of the longitude of the common node of the orbits in the usual development of the perturbative function. The above equations may then be put into the form

$$\frac{dR}{d\varphi'} = -\frac{m}{a'} \iota h \operatorname{cosec} \gamma \sin x' \sin N - \frac{m}{a'} (i'+j') h \tan \tfrac{1}{2}\gamma \sin x' \sin N - \tfrac{1}{2} \frac{m}{a'} \frac{dh}{du} \cos \tfrac{1}{2}\gamma \cos x' \cos N.$$

$$\frac{1}{\sin \varphi} \frac{dR}{d\theta'} = \frac{m}{a'} \iota h \operatorname{cosec} \gamma \cos x' \sin N + \frac{m}{a'} (i'+j') h \tan \tfrac{1}{2}\gamma \cos x' \sin N - \tfrac{1}{2} \frac{m}{a'} \frac{dh}{du} \cos \tfrac{1}{2}\gamma \sin x' \cos N.$$

Substituting these expressions in (6), and integrating, we shall have the values of $\delta\theta'$ and $\delta\phi'$, the perturbations of the inclination and node.

For the perturbations of the latitude, counted in the direction perpendicular to the plane of the orbit, we shall have

$$\begin{aligned} \delta\beta' &= \delta\phi' \sin (v'-\theta') - \sin \phi' \delta\theta' \cos (v'-\theta') \\ &= m\nu \sec \psi' \{T+I\} \sin (N+V) \\ &\quad + m\nu \sec \psi' \{T-I\} \sin (N-V) \end{aligned} \qquad (7)$$

Where

$$T = \tfrac{1}{4} \frac{dh}{du} \cos \tfrac{1}{2}\gamma; \quad I = \tfrac{1}{4} h \{ \iota \operatorname{cosec} \gamma + (i'+j') \tan \tfrac{1}{2}\gamma \}$$

$V =$ true distance of planet from common node.

Putting

$$B_1 = T+I; \qquad B_2 = T-I,$$

and developing V in terms of λ and f to terms of the second order with respect to the eccentricity, we shall have

$$\delta\beta = m\nu B_1 \left\{ \begin{array}{l} (1-e^2) \sin (N+\lambda) \\ + \ e \sin (N+\lambda+f) \\ - \ e \sin (N+\lambda-f) \\ + \tfrac{9}{8} e^2 \sin (N+\lambda+2f) \\ - \tfrac{1}{8} e^2 \sin (N+\lambda-2f) \end{array} \right\} + m\nu B_2 \left\{ \begin{array}{l} (1-e^2) \sin (N-\lambda) \\ + \ e \sin (N-\lambda-f) \\ - \ e \sin (N-\lambda+f) \\ + \tfrac{9}{8} e^2 \sin (N-\lambda-2f) \\ - \tfrac{1}{8} e^2 \sin (N-\lambda+2f) \end{array} \right\} \qquad (8)$$

For the perturbations of the constants which determine the position of the orbit, we put

$$p = \sin\phi \sin\theta; \qquad q = \sin\phi\cos\theta;$$

$\tau =$ longitude of common node of the two orbits.

We then have

$$\delta p' = 2\,m\nu\,\{I \sin\tau\cos N - T\cos\tau\sin N\};$$
$$\delta q' = 2\,m\nu\,\{I\cos\tau\cos N + T\sin\tau\sin N\}. \qquad (9)$$

Or,

$$\delta p' = \;\; m\nu\,\{(I-T)\sin(N+\tau) - (I+T)\sin(N-\tau)\};$$
$$\delta q' = \;\; m\nu\,\{(I-T)\cos(N+\tau) + (I+T)\cos(N-\tau)\};$$

§ 8. The equations (2) and (9) determine the periodic perturbations of the elements. For the secular variations, which proceed from those terms of the perturbative in which both i' and i are zero, the same expressions apply, only changing

$$\nu \sin N \text{ into } \;\; n't\cos N;$$
$$\nu \cos N \text{ into } - n't \sin N.$$

We therefore have, for the secular variations,

$$\frac{dl'}{dl} = \;\; mn' L_0 \cos N;$$
$$\frac{de'}{dt} = -mn' E_0 \sin N;$$
$$\frac{d\pi'}{dt} = \;\; mn' W_0 \cos N; \qquad (10)$$
$$\frac{dp'}{dt} = -2\,mn'\{I_0 \sin\tau\sin N + T_0\cos\tau\cos N\};$$
$$\frac{dq'}{dt} = -2\,mn'\{I_0\cos\tau\sin N - T_0\sin\tau\cos N\}.$$

Owing to the smallness of the eccentricity of Neptune, it will be advisable to substitute the rectangular co-ordinates of the centre of its orbit for the eccentricity and longitude of perihelion. The perihelion itself is subject to changes so great that it would otherwise be necessary to develop the perturbations to quantities of a higher order than the first. We shall, therefore, put

$$h = e\sin\pi; \qquad k = e\cos\pi.$$

For the secular variations of h and k, we then have, to a sufficient degree of approximation,

$$\left.\begin{aligned}\frac{dh}{dt} &= \;\; mn'e'W_0\cos(N+\pi');\\ \frac{dk}{dt} &= -mn'e'W_0\sin(N+\pi').\end{aligned}\right\} \qquad (11)$$

§ 9. *Development of the action of an inner on outer planet through the Sun.*

The perturbations which one planet produces on another may be divided into two distinct parts.

1. Those produced by their direct attraction on each other.

2. Those produced by the displacement of the Sun by the attraction of the disturbing planet. The co-ordinates of the disturbed planet being counted from the centre of the Sun, the displacement of the Sun not only changes the value of the co-ordinates by changing their origin, but also by modifying the attraction of the Sun itself.

The perturbations of both classes may be included in the same formulæ, and the total perturbations computed in the same way that those of the first class are, by a very simple modification of those functions of the ratio of the mean distances which enter into the different values of h. But in the case of the action of an inner on an outer planet more than twice as far from the Sun, this method will be subject to this serious inconvenience; that the perturbations of the elements are many times greater than those of the co-ordinates. Referring to formulæ (4) and (5), it will be remembered that L, E, and W really express perturbations of the mean longitude, perihelion, and eccentricity, and it will be seen that the perturbations of the true longitude δv are expressed as a function of the perturbations of those elements. Now, having in this way computed the perturbations of any co-ordinate which depend upon the different terms of the perturbative function, when we collect those coefficients which are multiplied by the sine or cosine of identical angles, we shall frequently find that their sum will nearly vanish, as has been already remarked. As this circumstance depends on a theorem of some importance, which will furnish a valuable check on the developments we shall presently give, it is worth while to trace it to its origin.

The elements of a planet depend on its *position* and its *velocity* at a given epoch; each element is a function of the co-ordinates, their differential coefficients, and the time, or, representing an element by a, and putting, for shortness,

$$\xi = \frac{dx}{dt}, \eta = \frac{dy}{dt}, \zeta = \frac{dz}{dt},$$

we have six equations of the form

$$a_n = f(x, y, z, \xi, \eta, \zeta, t) \tag{12}$$

When we express the co-ordinates as a function of the elements and the time, we have

$$x, y, \text{ or } z = f(a_1, a_2, a_3, a_4, a_5, a_6, t) \tag{13}$$

Substituting for the elements the values just given, ξ, η, and ζ must vanish identically in the value of each co-ordinate. If, now, the changes in ξ, η, and ζ are of a higher order of magnitude than those in x, y, and z, the co-ordinates will be subject to smaller variations than the elements.

Suppose, now, that one of the co-ordinates is affected with an inequality of which the period is very short compared with that of the revolution of the planet. Represent it by

$$c \sin(pnt + \varepsilon).$$

Its differential coefficient will be

$$pnc \cos(pnt + \varepsilon).$$

Since the *elements* contain this coefficient, and therefore include terms in which the large number p multiplies the coefficient of the angle, their perturbations will be much larger than that of the co-ordinate. But, in passing from the perturbations of the elements to those of the co-ordinates, these large terms must destroy each other.

Let us apply this principle to the case under consideration. That portion of the perturbative function which arises from the action of an inner planet on the Sun may be developed in a series of terms of the form

$$\frac{mc}{a'a^2} \cos (i'\lambda' + i\lambda + C);$$

c representing a number, not a line.

It therefore becomes infinite when a is infinitely small.

The second differential coefficient of the perturbation of any rectilineal co-ordinate of the outer planet will be of the order of magnitude

$$\frac{dR}{da'} = \frac{mc}{a^2} \cos N,$$

putting

$$N = i'\lambda' + i\lambda + C.$$

If we integrate this differential, and develop the quantity $\frac{c}{i'n' + in}$ according to the powers of $\frac{n'}{n} = \frac{a^{\frac{3}{2}}}{a'^{\frac{3}{2}}}$, the largest terms in the first differential coefficient of the co-ordinates will be of the form

$$\frac{mc}{ia^{\frac{1}{2}}} \sin N.$$

This also will become infinite when a is infinitely small; and since the perturbations of the elements contain these terms, it follows that they also will be infinite in this case. Finally, by another integration, we shall have for the largest perturbations of the co-ordinate itself

$$\frac{mca}{i^2} \cos N,$$

which will vanish when a is infinitely small. Hence, in the case under consideration, *although the perturbations of the elements become infinite, those of the co-ordinates vanish.*

The co-ordinates referred to are linear. The order of magnitude of the angular co-ordinates, or the logarithms of any linear co-ordinate, will be given by dividing by a'. We shall, therefore, have for largest term in the perturbations

$$\delta v,\ \delta\beta, \text{ or } \delta \log r = mca \begin{matrix}\sin\\ \cos\end{matrix} N.$$

Hence, when we collect the perturbations of the co-ordinates due to the cause in question, all terms of a higher order of magnitude than this ought to destroy each other identically.

§ 10. That portion of the perturbative function which is due to the action of the inner planet on the sun is

$$\frac{r'}{r^2}\cos V$$

V being the angular distance between the planets. Developing it in a series of terms of the form

$$\frac{mh}{a'}\cos(i'\lambda' + i\lambda + j'\omega' + j\omega)$$

h will be of the form $\frac{c}{a^2}$, c being a numerical coefficient, multiplied by powers of the eccentricities and mutual inclinations.

From this development, and the equations (3), (4), (5), (7), and (8), I have computed the following analytical values of the coefficients for the perturbations of the longitude, latitude, and logarithm of radius vector.

$$\begin{aligned}
\delta v &= \frac{m}{a^2}\Sigma\, V^{(i)} \sin N^{(i)} \\
\delta \log r &= \frac{m}{a^2}\Sigma\, R^{(i)} \cos N^{(i)} \\
\delta\beta &= \frac{m}{a^2}\Sigma\, B^{(i)} \sin N^{(a)}
\end{aligned} \qquad (16)$$

$$\begin{aligned}
V^{(1)} &= (1 - u^2 - \tfrac{1}{2}e^2)\,(3\nu_1^2 + 2\nu_1 + 3\nu_4 + \nu_5) \\
&\quad + e'^2\,(-\tfrac{3}{2}\nu_1^2 - \tfrac{1}{2}\nu_1 + 3\nu_5^2 - 2\nu_5 - \tfrac{5}{8}\nu_{12} + \tfrac{15}{8}\nu_{13}) \\
V^{(2)} &= ee'\,(6\nu_2^2 + \tfrac{3}{2}\nu_2 + 6\nu_3^2 - 3\nu_3 + 3\nu_8 + \tfrac{15}{2}\nu_{11}) \\
V^{(3)} &= e\,(-6\nu_3^2 + 4\nu_3 - 2\nu_2 - 6\nu_{11}) \\
V^{(4)} &= e'\,(-3\nu_1^2 - \tfrac{1}{2}\nu_1 - \tfrac{9}{4}\nu_4 - \tfrac{5}{4}\nu_5 + \tfrac{1}{2}\nu_{12}) \\
V^{(5)} &= e'\,(3\nu_1^2 + \tfrac{3}{2}\nu_1 + 3\nu_5^2 + \tfrac{3}{4}\nu_5 + \tfrac{15}{4}\nu_4 + \tfrac{3}{2}\nu_{13}) \\
V^{(9)} &= e^2\,(\tfrac{27}{8}\nu_6 - \tfrac{81}{8}\nu_9^2 + \tfrac{27}{4}\nu_9 - \tfrac{81}{8}\nu_{16}) \\
V^{(10)} &= e^2\,(-\tfrac{3}{8}\nu_7 + \tfrac{3}{8}\nu_{10}^2 + \tfrac{1}{4}\nu_{10} + \tfrac{1}{8}\nu_{17}) \\
V^{(11)} &= ee'\,(-\tfrac{5}{2}\nu_2 - 6\nu_3^2 + \nu_3 - \tfrac{9}{2}\nu_{11} + \nu_{18}) \\
V^{(12)} &= e'^2\,(-\tfrac{15}{4}\nu_1^2 - \tfrac{5}{8}\nu_1 - \tfrac{21}{8}\nu_4 - \tfrac{13}{8}\nu_5 + \tfrac{3}{8}\nu_{12}^2 + \tfrac{1}{8}\nu_{12} + \tfrac{1}{4}\nu_{19}) \\
V^{(13)} &= e'^2\,(\tfrac{15}{4}\nu_1^2 + \tfrac{15}{8}\nu_1 + \tfrac{39}{8}\nu_4 + 3\nu_5^2 + \tfrac{5}{8}\nu_5 + \tfrac{27}{8}\nu_{13}^2 + \tfrac{3}{8}\nu_{13} + 2\nu_{20}) \\
V^{(15)} &= u^2\,(3\nu_{15}^2 + 2\nu_{15} - 3\nu_{14} + \nu_{21})
\end{aligned} \qquad (17)$$

$$\begin{aligned}
R^{(1)} &= (1 - u^2 - \tfrac{1}{2}e^2)\,(-2\nu_1 - \tfrac{3}{2}\nu_4 + \tfrac{1}{2}\nu_5) \\
&\quad + e'^2\,(\tfrac{5}{4}\nu_1 + \tfrac{3}{4}\nu_4 + \tfrac{3}{2}\nu_5^2 - \tfrac{5}{4}\nu_5 + \tfrac{3}{8}\nu_{12} + \tfrac{9}{8}\nu_{13}) \\
R^{(2)} &= ee'\,(-\tfrac{9}{2}\nu_2 - 3\nu_3^2 + \tfrac{3}{2}\nu_3 + \tfrac{3}{2}\nu_8 - \tfrac{9}{2}\nu_{11}) \\
R^{(3)} &= e\,(\nu_2 + 4\nu_3 - 3\nu_{11}) \\
R^{(4)} &= e'\,(\tfrac{3}{2}\nu_1^2 + \tfrac{1}{4}\nu_1 + \tfrac{3}{4}\nu_4 + \tfrac{3}{4}\nu_5 + \tfrac{1}{4}\nu_{12}) \\
R^{(5)} &= e'\,(-\tfrac{3}{2}\nu_1^2 - \tfrac{3}{4}\nu_1 - \tfrac{9}{4}\nu_4 - \tfrac{9}{4}\nu_5 + \tfrac{3}{4}\nu_{13}) \\
R^{(9)} &= e^2\,(-\tfrac{27}{16}\nu_6 + \tfrac{27}{4}\nu^9 - \tfrac{81}{16}\nu_{16}) \\
R^{(10)} &= e^2\,(\tfrac{3}{16}\nu_7 - \tfrac{1}{4}\nu_{10} + \tfrac{1}{16}\nu_{17}) \\
R^{(11)} &= ee'\,(\tfrac{3}{2}\nu_2 + 3\nu_3^2 - \tfrac{1}{2}\nu_3 + \tfrac{3}{2}\nu_{11} + \tfrac{1}{2}\nu_{18}) \\
R^{(12)} &= e'^2\,(\tfrac{9}{4}\nu_1^2 + \tfrac{3}{8}\nu_1 + \tfrac{27}{16}\nu_4 + \tfrac{17}{16}\nu_5 - \tfrac{3}{8}\nu_{12} + \tfrac{1}{8}\nu_{19}) \\
R^{(13)} &= e'^2\,(-\tfrac{9}{4}\nu_1^2 - \tfrac{9}{8}\nu_1 - \tfrac{51}{16}\nu_4 - \tfrac{3}{2}\nu_5^2 - \tfrac{7}{16}\nu_5 - \tfrac{21}{8}\nu_{13} + \nu_{20}) \\
R^{(15)} &= u^2\,(\tfrac{3}{2}\nu^{14} - 2\nu_{15} + \tfrac{1}{2}\nu_{21})
\end{aligned} \qquad (18)$$

$$
\begin{aligned}
B^{(a)} &= u \;(-\;\; \nu_1 - \nu_{15})\\
B^{(b)} &= ue' \,(-\;\; \nu_1 - \tfrac{3}{2}\nu_4 + \;\; \nu_{15} - \tfrac{1}{2}\nu_{21})\\
B^{(c)} &= ue' \,(-\;\; \nu_1 + \tfrac{1}{2}\nu_5 - \tfrac{3}{2}\nu_{14} + \nu_{15})\\
B^{(d)} &= ue \;(\;\; 2\,\nu_3 - 2\,\nu_{22})
\end{aligned}
\qquad (19)
$$

The values of $N^{(i)}$ are as follows:

$$
\begin{array}{llll}
N^{(1)} = & \lambda' - \lambda & N^{(2)} = & 2\,\lambda' - 2\,\lambda - \omega' + \omega \\
N^{(3)} = & -\lambda' + 2\,\lambda - \omega & N^{(4)} = & \lambda - \omega' \\
N^{(5)} = & 2\,\lambda' - \lambda \;\; - \omega' & N^{(6)} = & -2\,\lambda' + 3\,\lambda + \omega' \;\; - 2\,\omega \\
N^{(7)} = & \lambda + \omega' \;\; - 2\,\omega & N^{(8)} = & 3\,\lambda' - 2\,\lambda - 2\,\omega' + \omega \\
N^{(9)} = & -\lambda' + 3\,\lambda - 2\,\omega & N^{(10)} = & \lambda' + \lambda \;\; - 2\,\omega \\
N^{(11)} = & 2\,\lambda - \omega \;\; - \omega' & N^{(12)} = & \lambda' + \lambda \;\; - 2\,\omega' \\
N^{(13)} = & 3\,\lambda' - \lambda \;\; - 2\,\omega' & N^{(14)} = & \lambda + \omega' \\
N^{(15)} = & \lambda' + \lambda & N^{(16)} = & 3\,\lambda - \omega' \;\; - 2\,\omega \\
N^{(17)} = & 2\,\lambda' + \lambda \;\; - \omega' - 2\,\omega & N^{(18)} = & \lambda' + 2\,\lambda - 2\,\omega' - \omega \\
N^{(19)} = & 2\,\lambda' + \lambda \;\; - 3\,\omega' & N^{(20)} = & 4\,\lambda' - \lambda \;\; - 3\,\omega' \\
N^{(21)} = & 2\,\lambda' + \lambda \;\; - \omega' & N^{(a)} = & \lambda \\
N^{(b)} = & \lambda' + \lambda \;\; - \omega' & N^{(c)} = & \lambda' - \lambda \;\; - \omega' \\
 & & N^{(d)} = & 2\,\lambda - \omega
\end{array}
\qquad (20)
$$

From these values of N the corresponding values of ν are derived, remembering that

$$\nu = \frac{n'}{i'n' + in}$$

i' and i being the coefficients of λ' and λ respectively in the value of N.

The check on the correctness of the preceding values of V, R, and B may now be applied by developing ν in powers of $\frac{n'}{n}$, and retaining only the first term; that is, by putting $\nu = \frac{1}{i}$, $\nu^2 = 0$. Making these substitutions, all the values of V, R, and B will be found to vanish. In other words, μ^2 will be the lowest power of μ which will enter into the values of V, R, or B, as we have already shown from *a priori* considerations.

For convenience, we shall give the values of V, R, and B developed according to the powers of μ, the ratio of the mean motions, a form similar to that in which the lunar inequalities are developed in the theory of the moon. Putting

$$\frac{i'}{i} = s,$$

we have

$$
\begin{aligned}
\nu &= \frac{\mu}{i}\{1 - \;\; s\mu + \;\; s^2\mu^2 - \;\; s^3\mu^3 + \text{etc.}\}\\
\nu^2 &= \frac{\mu^2}{i^2}\{1 - 2\,s\mu + 3\,s^2\mu^2 - 4\,s^3\mu^3 + \text{etc.}\}
\end{aligned}
$$

We shall also put

$$
\begin{aligned}
V_1 &= \frac{V}{a^3} = \frac{cV}{\mu^2} \\
R_1 &= \frac{R}{a^3} = \frac{cR}{\mu^2}, \\
B_1 &= \frac{B}{a^3} = \frac{cB}{\mu^2};
\end{aligned}
$$

c being a constant, equal to unity if we neglect the change of mean distance produced by the action of other planets. We then have

$$
\begin{aligned}
\delta v &= mac\Sigma V_1 \sin N, \\
\delta \log r &= mac\Sigma R_1 \cos N, \\
\delta \beta &= mac\Sigma B_1 \sin N.
\end{aligned}
\tag{21}
$$

Substituting the above developments for the ν's in V, R, and B, we have

$$
\begin{aligned}
V_1^{(1)} &= (1 - u^2 - \tfrac{1}{2} e^2)(-1 - \mu^2 - 6\mu^3 - 19\mu^4) \\
&\qquad + e'^2 (1 - 2\mu^2 - 30\mu^3) \\
V_1^{(2)} &= ee' (-\tfrac{3}{4} - \tfrac{3}{8}\mu^2 - \tfrac{27}{16}\mu^3) \\
V_1^{(3)} &= e\ (\tfrac{1}{2} + \tfrac{1}{8}\mu^2 + \tfrac{3}{8}\mu^3 + \tfrac{19}{32}\mu^4) \\
V_1^{(4)} &= e'\ (-\tfrac{1}{2} + \mu^2 + 9\mu^3 + 25\mu^4) \\
V_1^{(5)} &= e'\ (-\tfrac{3}{2} - 3\mu^2 - 27\mu^3 - 135\mu^4) \\
V_1^{(9)} &= e^2\ (\tfrac{3}{8} + \tfrac{1}{24}\mu^2 + \tfrac{1}{12}\mu^3) \\
V_1^{(10)} &= e^2\ (-\tfrac{1}{8} - \tfrac{1}{8}\mu^2 + \tfrac{3}{4}\mu^3) \\
V_1^{(11)} &= ee' (-\tfrac{1}{4} + \tfrac{1}{8}\mu^2 + \tfrac{9}{16}\mu^3) \\
V_1^{(12)} &= e'^2 (-\tfrac{1}{8} - 2\mu^2 - \tfrac{9}{4}\mu^3) \\
V_1^{(13)} &= e'^2 (-\tfrac{17}{8} - \tfrac{23}{8}\mu^2 - \tfrac{255}{4}\mu^3)
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
R_1^{(1)} &= (1 - u^2 - \tfrac{1}{2} e^2)(1 - 2\mu^2 - 6\mu^3 - 14\mu^4 - 30\mu^5) \\
&\qquad + e'^2 (-1 - 4\mu^2 - 24\mu^3) \\
R_1^{(2)} &= ee' (\tfrac{3}{4} - \tfrac{3}{4}\mu^2 - \tfrac{15}{8}\mu^3) \\
R_1^{(3)} &= e\ (\tfrac{1}{2} - \tfrac{1}{4}\mu^2 - \tfrac{3}{8}\mu^3 - \tfrac{7}{16}\mu^4) \\
R_1^{(4)} &= e'\ (-\tfrac{1}{2} - 2\mu^2 - 6\mu^3 - 17\mu^4) \\
R_1^{(5)} &= e'\ (\tfrac{3}{2} - 6\mu^2 - 30\mu^3 - 116\mu^4) \\
R_1^{(9)} &= e^2\ (\tfrac{3}{8} - \tfrac{1}{12}\mu^2 - \tfrac{1}{12}\mu^3) \\
R_1^{(10)} &= e^2\ (\tfrac{1}{8} - \tfrac{1}{4}\mu^2 + \tfrac{3}{4}\mu^3) \\
R_1^{(11)} &= ee' (-\tfrac{1}{4} - \tfrac{1}{4}\mu^2 - \tfrac{3}{8}\mu^3) \\
R_1^{(12)} &= e'^2 (-\tfrac{1}{8} - \tfrac{1}{2}\mu^2 + \tfrac{9}{4}\mu^3) \\
R_1^{(13)} &= e'^2 (\tfrac{17}{8} - \tfrac{31}{2}\mu^2 - \tfrac{405}{4}\mu^3) \\
R_1^{(15)} &= u^2 (1 - 2\mu^2 + 14\mu^3)
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
B_1^{(a)} &= u\ (2 + 2\mu^2 + 2\mu^4) \\
B_1^{(b)} &= e'u\ (1 + 4\mu^2 - 6\mu^3) \\
B_1^{(c)} &= e'u\ (1 + 4\mu^2 - 6\mu^3) \\
B_1^{(d)} &= eu\ (1 + \tfrac{1}{4}\mu^2 + \tfrac{1}{16}\mu^4)
\end{aligned}
\tag{24}
$$

Such are the formulæ by which we shall proceed to compute the perturbations of Neptune by Jupiter, Saturn, and Uranus.

It will be noticed that the coefficient of μ vanishes identically in the last developments. I have not completely investigated this law, but it seems to arise from the circumstance that that portion of the perturbation in question which proceeds from the change in the origin of co-ordinates is independent of μ, while that portion which is caused by the modified attraction of the Sun is of the order of magnitude μ^2. It furnishes a yet more valuable check than the last on the developments.

§ 11. Allusion has already been made to the complications introduced into the theory of Neptune by the near approach of its mean motion to double that of Uranus, and the consequent oscillation of all the elements of its orbit in a cycle of 4300 years of duration. In order to construct a dynamical theory which should be correct within a tenth of a second through the whole of one of these cycles, it would be necessary to include many terms dependent on the second, and perhaps some dependent on the third power of the masses of the disturbing planets.

If this task were accomplished, the necessary uncertainty in the mass of Uranus and the elements of Neptune would destroy all the value of the theory. A change of one-tenth in the mass of Uranus would produce a change of 200″ in the coefficient of the perturbation of the mean longitude. The mean motions of Walker and Kowalski being each about 8″ in error, the place of the planet from this cause alone would be in error by nearly 10° at the end of a cycle.

After much careful consideration of different ways of relieving the theories of Uranus and Neptune from the complexities introduced by the large perturbations referred to, I finally determined to develop them not as perturbations of the coordinates, but of the elements. It will readily be seen that if the eccentricity or perihelion is greater than the mean during several revolutions of the planet, there will be a perturbation in the radius vector and longitude having nearly the same period with the revolution of the planet, although the latter may really scarcely wander from a true elliptic orbit during an entire revolution. In such a case it is clearly best, in constructing a theory designed to remain of the highest degree of exactness for only a few centuries, to take not the mean values of the elements, but their values at a particular epoch during the time the theory is expected to be used.

In doing this, we shall be treating the change in the elements in the same way that the secular variations are usually treated. These variations are really periodic, and in a perfect theory would have to be treated as such. But the elliptic elements on which all our planetary theories are founded are not mean elements, but elements brought up by secular variation to the epoch 1800 or 1850.

Thus, our perturbations of the elements will be of the form

$$\delta a = c + a_1 t + \Sigma a_2 \begin{matrix}\sin\\ \cos\end{matrix} \{kt + \varepsilon\},$$

in which a' is the secular variation proper, k a small coefficient equal to $2n' - n$ or its multiples, and c a constant added to the integral, of such value as to make δa vanish at the epoch 1850.

§ 12. *Adopted elements and masses.*

The elements of Neptune adopted in the computation of the perturbations are obtained by correcting those of Walker so as to agree with the Lalande observations, and as nearly as possible with seven normal places derived from the modern observations from 1846 to 1863. The latter series is thus represented within a second of arc. As these elements are merely provisional, it is not worth while to give any details of the corrections, except their amounts, which are as follows:

$$\begin{array}{llrl}
\delta\pi = -4^\circ\, 11'\, 18''.6; & \pi & = & 43^\circ\, 3'\, 18''.6 \\
\delta e = -.00025451; & e & = & .00846495 \\
\delta n = -8''.406; & n & = & 7864''.368 \\
\delta\varepsilon = -3'\, 5''.92; & \varepsilon & = & 335\ 5\ 31.10 \\
 & \log a & = & 1.4780405 \\
 & i & & 1^\circ\, 47'\, 1'' \\
 & \Omega & & 130\ \ 7\ 20
\end{array}$$

Epoch, Jan. 0, 1850, Greenwich, M. noon.

To obtain the value of log a, the mean motion was diminished by the secular variation of the longitude of the epoch $= 21''.354$. A more exact value of this quantity will appear, in the course of our computations, to be $21''.4426$.

The provisional inclination and longitude have been taken from Walker without change, as the small corrections which his values of these elements may require will not affect the perturbations.

The adopted elements of Uranus, Saturn, and Jupiter, with their functions used in the theory for the same epoch, are as follows:

	Uranus.	Saturn.	Jupiter.
π	167° 34′ 21″	90° 4′ 0″	11° 54′ 51″
ε	28 27 14	14 48 40	159 56 20
i	0 46 30	2 29 28.8	1 18 41.1
θ	73 14 14	112 22 14	98 56 10
n	15425.030	43996.127	109256.72
e —	.0466972	.0560050	.0482273
$\log a$	1.2837047	0.9802225	0.7162201
τ	335°38′	77°56′	355°52′
u	.0131517	.0083880	.0082735
α	0.638195	0.317301	0.1727703
m	$\frac{1}{21000}$	$\frac{1}{3501.6}$	$\frac{1}{1047.879}$

These elements of Uranus have been obtained by applying to Peirce's values of the mean elements (Appendix to American Ephemeris and Nautical Almanac, 1860–64, p. 4) approximate long-period perturbations of the elements produced by Neptune at the epoch 1850. The elements of Jupiter and Saturn are from Hansen's prize memoir on the mutual perturbations of those planets, and are, substantially, the same as Bouvard's.

The values of those constants which depend on the ratio of the mean distances are as follows, using the notation of the Mécanique Céleste:

I.—URANUS AND NEPTUNE.

i	$b_{\frac{1}{2}}^{(i)}$	$a\frac{db_{\frac{1}{2}}^{(i)}}{da}$	$a^2\frac{d^2b_{\frac{1}{2}}^{(i)}}{da^2}$	$a^3\frac{d^3b_{\frac{1}{2}}^{(i)}}{da^3}$	$a^4\frac{d^4b_{\frac{1}{2}}^{(i)}}{da^4}$
0	2.26969	0.72903	1.8326	6.4384	35.17
1	—1.68379	6.05279	—13.0023	65.5556	—259.42
2	0.37751	0.95867	2.1283	6.7135	35.99
3	0.20310	0.72530	2.2389	7.4924	36.95
4	0.11422	0.52446	2.1024	8.2270	39.52
5	0.06593	0.36954	1.8319	8.5192	43.00
6	0.03870	0.25606	1.5157	8.302	46.01
7	0.02299	0.17533	1.2085	7.679	47.57
8	0.01379	0.11900	0.9365	6.804	47.27
9	0.00832	0.0802	0.7100	5.818	45.18
10	0.0051	0.054	0.533		

i	$ab_{\frac{3}{2}}^{(i)}$	$a^2\frac{db_{\frac{3}{2}}^{(i)}}{da}$	$a^3\frac{d^2b_{\frac{3}{2}}^{(i)}}{da^2}$
0	—0.8966	26.5493	2.80
1	3.2907	11.9366	60.92
2	2.4710	11.0760	59.76
3	1.7806	9.6115	57.07
4	1.2524	7.9427	52.59
5	0.8668	6.3301	46.80
6	0.5931	4.9065	40.34
7	0.4023	3.7215	33.83
8	0.2711	2.7738	27.69
9	0.1817	2.0381	22.20

It will be observed that in $b_{\frac{1}{2}}^{(1)}$, $ab_{\frac{3}{2}}^{(0)}$, and their differential coefficients, we have included those multiples of $\frac{1}{\alpha^2}$ which are introduced by the action of Uranus on the Sun. It seemed less laborious to do this than to make a separate computation of the terms produced by this cause. But for Saturn and Jupiter $\frac{1}{\alpha^2}$ is so large that it will be better to use the developments previously given.

II.—SATURN AND NEPTUNE.

i	$b_{\frac{1}{2}}^{(i)}$	$a\frac{db_{\frac{1}{2}}^{(i)}}{da}$	$a^2\frac{d^2b_{\frac{1}{2}}^{(i)}}{da^2}$	$a^3\frac{d^3b_{\frac{1}{2}}^{(i)}}{da^3}$
0	2.05341	0.11342	0.14186	0.0986
1	0.33010	.35745	.08964	.1313
2	.07890	.16509	.19632	.1075
3	.02091	.06476	.14027	.1878
4	.00581	.02383	.07517	.1701
5	.00166	.00847	.03514	

i	$ab_{\frac{3}{2}}^{(i)}$	$a^2\frac{db_{\frac{3}{2}}^{(i)}}{da}$
0	0.8045	0.4003
1	0.3686	.5242
2	.1443	.3456
3	.0531	.1794
4	.0189	
5	0066	

III.—JUPITER AND NEPTUNE.

i	$b_{\frac{1}{2}}^{(i)}$	$a\frac{db_{\frac{1}{2}}^{(i)}}{da}$	$a^2\frac{d^2b_{\frac{1}{2}}^{(i)}}{da^2}$	$a^3\frac{d^3b_{\frac{1}{2}}^{(i)}}{da^3}$
0	2.01518	0.03088	0.0330	0.0067
1	0.17474	.17876	.0124	.0139
2	.02267	.04592	.0483	.0074
3	.00327	.00989	.0202	.0221
4	.00049	.00199	.0061	.0125

	$ab_{\frac{3}{2}}^{(i)}$	$a^2\frac{db_{\frac{3}{2}}^{(i)}}{da}$
0	0.3699	0.4209
1	.0948	.2005
2	.0204	.0634
3	.0041	.0168
4	0.0008	

§ 13. From these data the coefficients h of the different terms of the perturbative function, their differential coefficients, and the perturbations of the coordinates, are found to be as in the following table. The N's, it will be seen, are grouped according to the values of their constant parts, $j'\omega' + j\omega$.

h, its differential coefficients, L, W, and E, are given in units of the *third place* of decimals, to avoid writing zeros. The logarithms are reduced to the common base, 10, and are expressed in units of the seventh place.

ACTION OF URANUS.

I.—$j=0$; $j'=0$; $\iota=0$.

i'	0	+1	2	3	4	5	6	7	8	9
i	0	−1	−2	−3	−4	−5	−6	−7	−8	−9
h	+1135.50	−840.31	+187.98	+100.37	+55.83	+31.75	+18.29	+10.63	+6.2	+3.7
$aDah$	+367.47	+3025.10	+478.7	+359.3	+256.9	+178.5	+121.4	+81.2	+53.6	+36
$De'h$	+6.96	+12.52	−4.1	−7.6	−8.7	−8.4	−7.4	−6.2	−5.0	−4
$\frac{1}{2}Duh$	−10.84	−2.56	−8.35	−6.1	−4.3	−3.0	−2.1	−1.4	−1.0	
L	+3005.97	+1747.45	+1919.99	+1232.4	+799.6	+519.5	+336.3	+216.9	+138.9	+91
$e'W$	6.96	+12.52	−4.1	−7.6	−8.7	−8.4	−7.4	−6.2	−5.0	
E	0	+3.56	−1.6	−1.3	−0.9	−0.7	−0.5	−0.3	−0.2	
		″	″	″	″	″	″	″	″	″
$\delta l \div \sin N$		−17.853	−9.808	−4.197	−2.043	−1.061	−0.572	−0.317	−0.178	−0.102
$\delta v \div \sin(N-f)$		+0.013	−0.112	−0.066	−0.042	−0.028	−0.018	−0.012	−0.008	
$N+f$		−0.060	−0.096	−0.057	−0.037	−0.025	−0.017	−0.011	−0.008	
$N-2f$		+0.000	−0.001	−0.001						
$N+2f$		−0.001	−0.001	−0.001						
$\delta \log r \div \cos N$		+361	−81	−43	−24	−14	−8	−5	−3	−2
$N-f$		0	−1	−1	0	0	0	0	0	0
$N+f$		0	+1	+1	0	0	0	0	0	0
		″	″	″	″	″	″	″	″	
$\delta\beta \div \sin(N-V)$		−0.030	+0.055	+0.028	+0.014	+0.008	+0.005	+0.003	+0.002	
$(N+V)$		+0.083	+0.030	+0.014	+0.007	+0.004	+0.002	+0.001	+0.001	

II.—$j=1$; $j'=-1$; $\iota=0$.

i'	−7	−6	−5	−4	−3	−2	−1	0	+1	2	3	4	5	6	7
i	+7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7
h	+0.21	+0.26	+0.30	+0.33	+0.32	+0.24	+0.05	−0.24	−0.32	−0.91	+0.35	+0.48	+0.50	+0.46	+0.40
$aDah$	+1.8	+1.9	+1.9	+1.65	+1.16	+0.38	−0.56	−1.32	−1.50	+1.05	+0.20	+1.24	+2.00	+2.37	+2.5
$De'h$	+24.8	+30.5	+35.1	+38.5	+37.4	+27.93	+5.69	−28.77	−38.29	−107.13	+40.77	+56.1	+58.0	+53.0	+44.5
L	+4.8	+5.2	+5.4	+5.1	+4.1	+2.10	−0.82	−2.24	−4.80	−3.01	+2.4	+5.1	+6.8	+7.3	+7.2
$e'W$	+24.8	+30.5	+35.1	+38.5	+37.4	+27.93	+5.69	−28.77	−38.29	−107.13	+40.77	+56.1	+58.0	+53.0	+44.5
E	+24.6	+30.3	+34.9	+38.3	+37.2	+27.81	+5.55	−28.95	−38.47	−107.20	+40.61	+55.9	+57.6	+52.4	+43.7
	″	″	″	″	″	″	″		″	″	″	″	″	″	″
$\delta l \div \sin N$	+0.007	+0.009	+0.011	+0.013	+0.014	+0.011	−0.008		+0.049	+0.015	−0.008	−0.013	−0.014	−0.012	−0.010
$\delta v \div \sin(N-f)$	−0.072	−0.103	−0.143	−0.196	−0.255	−0.284	−0.115		−0.783	−1.095	+0.277	+0.286	+0.236	+0.179	+0.130
$N-2f$	−0.001	−0.001	−0.002	−0.002	−0.003	−0.003	−0.001		−0.008	−0.012	+0.003	+0.003	+0.002	+0.002	+0.001
$N+f$							−0.002		+0.002						
$\delta \log r \div \cos N-f$	−1	−1	−1	−2	−3	−3	−1		−8	−11	+3	+3	+2	+2	+1

III.—$j=-1$; $j'=0$; $\iota=0$.

i'	−5	−4	−3	−2	−1	0	+1	2	3	4	5	6	7	8
i	+6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7
h	+6.5	+9.0	+11.4	+12.7	−219.7	−17.00	−62.71	−57.573	−45.20	−33.27	−23.65	−16.41	−11.20	−7.6
$aDah$	+34.1	+36.0	+32.1	+17.4	+444.1	−59.5	−120.55	−161.57	−170.23	−157.9	−135.8	−110.3	−86.4	−66.2
$De'h$	−2.0	−1.7	−1.3	−1.0	+1.01	−1.50	−1.29	+0.61	+2.96	+4.85	+6.0	+6.4	+6.2	+5.7
$\frac{1}{2}Duh$	0	0	0	+0.5	+2.75	+2.31	+3.23	+3.61	+3.48	+3.09	+2.6	+2.2	+1.4	+1.2
L	+95.5	+108.6	+108.7	+79.8	+223.3	−153.0	−178.40	+8506.72	−871.70	−594.2	−443.5	−331.0	−244.5	−180
$e'W$	−2.0	−1.7	−1.3	−1.0	+1.01	−1.50	−1.29	+0.61	+2.96	+4.85	+6.0	+6.4	+6.2	+5.7
E	+0.1	+0.2	+0.2	+0.1	−0.95	0.00	+0.27	+0.49	+0.57	+0.56	+0.5	+0.4	+0.3	+0.3
	″	″	″	″	″	″	″		″	″		″	″	″
$\delta l \div \sin N$	+0.138	+0.184	+0.220	+0.202	+0.750	−0.766	−1.752	+2163.605	+0.279	+3.098	+1.531	+0.854	+0.504	+0.308
$\delta v \div \sin(N-f)$	+0.004	+0.004	+0.004	+0.004	+0.006	+0.001	−0.005		+0.116	+0.054	+0.035	+0.025	+0.018	+0.013
$(N+f)$	+0.004	+0.005	+0.005	+0.005	0	+0.001	0		+0.104	+0.049	+0.032	+0.022	+0.016	+0.012
$\delta \log r \div \cos N$	−2	−2	−3	−3	+31	0	−26		+61	+29	+17	+11	+6	+4
$(N-f)$									+1	+1				
$(N+f)$									−1	−1				
				″	″	″	″		″	″	″	″	″	″
$\delta\beta \div \sin(N-V)$	0	0	0	+0.002	+0.004	+0.012	+0.036		−0.046	−0.021	−0.011	−0.007	−0.004	−0.003
$\sin(N+V)$				+0.001	+0.014	+0.012	+0.028		−0.028	−0.011	−0.006	−0.004	−0.002	−0.001

IV.—$j=0$; $j'=-1$; $\iota=0$.

i'	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9
i	+6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
h	−0.68	−0.91	−1.13	−1.21	−0.72	+32.75	+12.73	+4.267	+12.016	+0.00	+6.43	+4.47	+3.04	+2.05	+1.35
$aDah$	−4.17	−4.56	−4.29	−2.71	+0.97	−54.83	+14.12	+47.61	+33.38	+33.80	+30.55	+25.64	+20.5	+15.8	+11.8
$De'h$	−79.9	−107.5	−133.8	−142.5	−85.17	+3805.68	+1503.24	+504.23	+1418.28	+1061.5	+758.9	+527.1	+358.7	+240.1	+158.8
$\frac{1}{4}Duh$	0	0	0	−0.1	−0.16	−0.52	−0.51	−0.611	−0.72	−0.68	−0.6	−0.5	−0.4	−0.3	0
L	−11.5	−13.2	−13.5	−10.3	−0.62	−27.80	+21.87	−557.07	+214.18	+147.4	+111.0	+83.5	+62.0	+45.3	+32.1
E	−80.1	−107.5	−133.9	−142.6	−85.24	+3865.40	+1502.86	+504.02	+1418.32	+1061.7	+759.4	+527.7	+359.3	+240.7	+159.4
	″	″	″	″	″	″	″	″	″	″					
$\delta l \div \sin N$	−0.017	−0.022	−0.027	−0.026	−0.002	−0.139	+0.215	−141.69	−2.279	−0.770	−0.383	−0.216	−0.128	−0.077	−0.046
$\delta v \div \sin(N-f)$	+0.232	+0.364	+0.543	+0.721	+0.573	−38.716	−29.524		+30.174	+11.061	+5.239	+2.720	+1.478	+0.823	+0.467
$N-2f$	+0.002	+0.004	+0.006	+0.008	+0.006	−0.411	−0.313		+0.321	+0.117	+0.055	+0.029	+0.016	+0.009	+0.005
$N-3f$	0	0	0	0	0	−0.005	−0.003		+0.004	+0.001	+0.001	0	0	0	0
$N+f$	0	0	0	0	0	−0.002	−0.001		−0.011	−0.008	−0.005	−0.003	−0.002	−0.002	−0.001
$\delta \log r \div \cos N$	0	0	0	0	0	+2	+6		−17	−8	−5	−3	−2	−1	0
$\div \cos(N-f)$	+2	+4	+6	+8	+6	−408	−311		+318	+116	+55	+29	+15	+9	+6
$N-2f$	0	0	0	0	0	−5	−4		+4	+1	+1	0	0	0	0
						″	″		″	″	″	″			
$\delta\beta \div \sin(N+V)$	0	0	0	0	0	−0.003	−0.005		−0.005	−0.002	−0.001	−0.001	0	0	0
$(N-V)$	0	0	0	0	0	−0.003	−0.005		−0.010	−0.005	−0.003	−0.002	−0.001	0	0

V.—$j=-2$; $j'=1$; $\iota=0$.

i'	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8
i	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7
h	+0.03	+0.02	−0.07	+0.01	+0.02	+0.006	−0.0145	−0.033	−0.016	−0.055	−0.06	−0.05	−0.05
$aDah$	+0.1	+0.0	+0.2	0.0	+0.08	+0.08	+0.014	−0.11	−0.25	−0.34	−0.40	−0.44	−0.44
$2De'h$	+6.5	+4.6	−15.9	+1.4	+5.1	+1.3	−3.87	−7.9	−11.0	−13.0	−13	−12	−11
	″				″	″	″	″	″	″	″	″	″
$\delta l \div \sin N$	+0.001	0	0	0	+0.001	+0.002	+0.56	+0.007	+0.005	+0.004	+0.003	+0.002	+0.002
$\delta v \div \sin(N+f)$	−0.011	−0.009	+0.040	−0.005	−0.026	−0.013		−0.084	−0.057	−0.044	−0.032	−0.024	−0.019

VI.—$j=1$; $j'=-2$; $\iota=0$.

i'	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8
i	+5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7
h	−.002	−.002	−.001	−.001	−.002	−.004	−.0058	−.007	+.006	+.012	+.016	+.018	+.018
$2De'h$	−1.0	−0.9	−0.7	−0.65	−1.04	−2.10	−2.80	−3.47	+2.86	+5.9	+7.8	+8.6	+8.6
							″	″					
δl	0	0	0	0	0	0	+0.21	−0.001	0	0	0	0	0
	″	″	″	″	″	″		″	″	″	″	″	″
$\delta v + \sin(N-f)$	+0.002	+0.002	+0.002	+0.002	+0.005	+0.021		−0.037	+0.015	+0.020	+0.020	+0.017	+0.015

VII.—$j=1$; $j'=0$; $\iota=-2$.

i'	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8
i	+5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7
h							−0.031						
$\frac{1}{4}Duh$	−1.3	−1.7	−1.9	−2.2	−2.2	−1.85	−1.17	−0.66	−0.3	−0.1	0	0	0
							″						
$\delta l + \sin N$							+1.11						
	″	″	″	″	″	″	″	″	″				
$\delta\beta + \sin(N-V)$	−0.004	−0.007	−0.010	−0.015	−0.022	−0.036	−0.58	+0.007	+0.002	0	0	0	0

VIII.—$j = 0;\ j' = 1;\ \iota = -2.$

i' i	+4 +5	−3 4	−2 3	−1 2	0 1	1 0	2 −1	3 −2	4 −3	5 −4	6 −5	7 −6
$2De'h$	+1.6	+1.8	+1.8	+1.6	+2.3	+0.4	+0.10	−0.09	−0.18	−0.21	−0.17	−0.14
$\frac{1}{2}Duh$	+0.5	+0.6	+0.6	+0.5	+0.75	+0.15	+0.033	−0.03	−0.06	−0.00	−0.04	0
	″	″	″	″	″	″		″	″	″		
$\delta v \div \sin(N+f)$	−0.003	−0.004	−0.005	−0.005	−0.012	−0.004		−0.001	−0.001	−0.001	0	0
	″	″	″	″	″	″						
$\delta\beta \div \sin(N-V)$	+0.001	+0.001	+0.002	+0.002	+0.004	+0.001		0	0	0	0	0

IX.—$j = -2;\ j' = 0;\ \iota = 0.$

i' i	−3 −5	−2 4	−1 3	0 2	1 1	2 0	3 −1	4 −2	5 −3	6 −4	7 −5	8 −6	9 −7
h	+0.66	+0.64	−17.57	+0.10	+0.214	+2.766	+3.750	+3.944	+3.660	+3.151	+2.58	+2.01	+1.57
$aDah$	+2.07	+1.36	+39.8	+1.36	+5.03	+9.55	+15.48	+19.71	+21.67	+21.6	+20.2	+18.0	+15.4
$De'h$	−0.06	−0.01	+0.19	+0.07	+0.12	+0.06	−0.17	−0.51	−0.86	−1.15	−1.35	−1.4	−1.4
L	+6.4	+4.7	+33.7	+2.9	+10.3	+16.3	+5.32	−565.6	+112.77	+80.22	+64.9	+53.1	+41.9
	″	″	″	″	″	″	″	″	″	″	″	″	″
$\delta l \div \sin N$	+0.009	+0.008	+0.068	+0.007	+0.034	+0.080	+0.050	−71.93	−1.253	−0.425	−0.227	−0.138	−0.087
$\delta v \div \sin(N-f)$	0	0	0	0	0	0	+0.002	0	−0.021	−0.010	−0.007	−0.005	−0.003
$(N+f)$	0	0	0	0	0	0	+0.001	0	−0.018	−0.009	−0.007	−0.005	−0.003

X.—$j = -1;\ j' = -1.$

i' i	−3 +5	−2 4	−1 3	0 2	1 1	2 0	3 −1	4 −2	5 −3	6 −4	7 −5	8 −6	9 −7
h	−0.113	−0.107	−0.090	+2.814	−0.325	−1.305	−1.903	−2.067	−1.954	−1.700	−1.40	−1.12	−0.86
$aDah$	−0.50	−0.42	−0.40	−6.48	−1.50	−3.38	−6.03	−8.30	−9.63	−10.01	−9.62	−8.74	−7.58
$De'h$	−13.3	−12.5	−10.5	+331.8	−38.3	−154.2	−224.67	−243.19	−228.57	−197.7	−161.6	−127.3	−96.9
L	−1.4	−1.2	−1.1	−5.9	−3.5	−6.11	−0.38	+29.95	−57.29	−40.85	−33.2	−27.4	−22.3
E	−13.3	−12.5	−10.5	+331.8	−38.7	−154.2	−224.65	−243.20	−228.68	−197.8	−161.8	−127.5	−97.0
	″	″	″	″	″	″	″	″	″	″	″	″	″
$\delta l \div \sin N$	−0.002	−0.002	−0.002	−0.015	−0.011	−0.030	−0.003	+38.09	+0.636	+0.217	+0.116	+0.065	+0.042
$\delta v \div \sin(N-f)$	+0.038	+0.012	+0.012	−1.662	+0.257	+1.514	+4.250		−5.074	−2.102	−1.131	−0.664	−0.402
$(N-2f)$	0	0	0	−0.017	+0.003	+0.016	+0.645		−0.054	−0.022	−0.012	−0.007	−0.004
$\delta \log r \div \cos N$	0	0	0	0	0	0	−2		+5	+2	+1	+1	0
$(N-f)$	0	0	0	−17	+3	+16	+44		−53	−22	−12	−7	−4

XI.—$j = 0;\ j' = -2.$

i' i	−3 +5	−2 4	−1 3	0 2	1 1	2 0	3 −1	4 −2	5 −3	6 −4	7 −5	8 −6	9 −7
h	0	0	0	+.01	+0.01	+0.14	+0.169	+0.2677	+0.259	+0.228	+0.206	+0.152	+0.118
$aDah$	+.03	+.03	+.03	+0.06	+0.18	+0.25	+0.655	+0.810	+1.018	+1.12	+1.11	+1.04	+0.92
$De'h$	+1.1	+1.1	+1.1	+2.08	+1.76	+32.51	+40.15	+63.13	+60 52	+52.90	+42.98	+34.1	+25.8
L					+0.4	+0.50	+0.36	−39.2	+7.24	+5.15	+4.4	+3.5	+2.2
E	+1.1	+1.1	+1.1	+2.1	+1.8	+32.50	+40.15	+63.13	+60.54	+52.92	+43.01	+34.1	+25.8
					″	″	″	″	″	″	″	″	″
$\delta l \div \sin N$	0	0	0	0	+0.001	+0.002	+0.003	−4 99	−0.080	−0.027	−0.015	−0.009	−0.005
	″	″	″	″	″	″	″		″	″	″	″	″
$\delta v \div \sin(N-f)$	−0.003	−0.004	−0.004	−0.010	−0.012	−0.319	−0.757		+1.344	+0.564	+0.301	+0.178	+0.110
$(N-2f)$	0	0	0	0	0	−0.003	−0.008		+0.014	+0.006	+0.003	+0.002	+0.001
$\delta \log r \div \cos(N-f)$	0	0	0	0	0	−3	−8		+15	+6	+4	+2	+1

XII.—$j = -3$; $j' = 0$.

i'	6
i	−3
L	+51.5
$\delta l \div \sin N$	″ +4.37

XIII.—$j = -2$; $j' = -1$.

i'	0	1	2	3	4	5	6	7	8	0	10
i	+3	2	1	0	−1	−2	−3	−4	−5	−6	−7
L							−38.5				
$2De'h$	+54.5	+1.4	+6.0	+23.4	+41.7	+55.17		+62.63	+58.87	+53.4	+44.8
$\delta l \div \sin N$							−3.26				
$\delta v \div \sin (N - f)$	″ −0.091	″ −0.003	″ −0.015	″ −0.076	″ −0.201	″ −0.503		″ +0.727	″ +0.320	″ +0.189	″ +0.116
$N - 2f$				−0.001	−0.002	−0.005		+0.008	+0.003	+0.002	+0.001

XIV.—$j = -1$; $j' = -2$.

i'	0	1	2	3	4	5	6	7	8	9	10
i	+3	2	1	0	−1	−2	−3	−4	−5	−6	−7
L							+10.0				
$2De'h$	−0.3	−1.8	−2.8	−10.9	−20.7	−28.4		−33.30	−31.7	−28.5	−24.6
$\div \sin N$							″ +0.85				
$\delta v \div \sin (N - f)$	0	″ +0.004	″ +0.007	″ +0.036	″ +0.100	″ +0.259		″ +0.387	″ +0.172	″ +0.101	″ −0.065

XV.—$j = 0$; $j' = -3$.

i'	0	1	2	3	4	5	6	7	8	9	10
i	+3	2	1	0	−1	−2	−3	−4	−5	−6	−7
$2De'h$	0	+0.1	+0.3	+1.2	+2.3	+3.0	+4.25	+4.47	+4.24	+3.8	+3.3
$\delta v \div \sin (N - f)$	0	0	″ −0.001	″ −0.004	″ −0.011	″ −0.033		″ +0.052	″ +0.023	″ +0.013	″ +0.009

ACTION OF SATURN.

I.—$j=0$; $j'=0$.

i'	0	1	2	3	4
i	0	−1	−2	−3	−4
h	+1026.84	+164.82	+39.15	+10.3	+2.8
$aDah$	+57.04	+178.60	+82.47	+32.3	+11.9
$De'h$	+0.780	−1.092	−1.56	−1.02	−0.5
$\frac{1}{2}Duh$	−0.773	−0.99	−0.44	−0.17	−0.06
L	+2167.76	+794.46	+268.80	+91.9	+31.2
E	0	−0.70	−0.33	−0.1	−0.1
		″	″	″	″
$\delta l \div \sin N$		−10.186	−1.723	−0.394	−0.100
$\delta v \div \sin (N-f)$		−0.109	−0.027	−0.008	−0.003
$(N+f)$		−0.091	−0.022	−0.007	−0.003
$\delta \log r \div \cos N$		−89	−21	−6	−2
		″	″		
$\delta\beta \div \sin (N+V)$		+0.004	+0.001	″	
$N-V$		+0.022	+0.005	+0.001	

II.—$j=1$; $j'=-1$.

i'	−3	−2	−1	0	1	2	3
i	+3	2	1	0	−1	−2	−3
h	+0.02	+0.04	+0.05	−0.0171	−0.044	+0.139	+0.124
$aDah$	+0.09	+0.13	+0.09	−0.058	−0.11	+0.12	+0.25
$De'h$	+2.8	+5.0	+5.89	−2.020	−5.17	+16.5	+14.7
L	+0.24	+0.39	+0.33	−0.159	−0.36	+0.68	+0.89
	″	″	″		″	″	″
$\delta l \div \sin N$	+0.001	+0.002	+0.004		+0.005	−0.004	−0.004
	″						
$\delta v \div \sin (N-f)$	−0.024	−0.064	−0.151		−0.133	+0.211	+0.124
$N-2f$	0	−0.001	−0.002		−0.001	+0.002	+0.001
$\delta \log r \div \cos (N-f)$	0	−1	−2		+1	0	0

III—$j=-1$; $j'=0$.

i'	−2	−1	0	1	2	3	4
i	+3	2	1	0	−1	−2	−3
h	+4.23	+8.48	−3.175	−28.49	−13.47	−5.32	−1.96
$aDah$	+8.4	+7.5	−7.14	−32.54	−28.59	−16.5	−8.1
$De'h$	−0.2	−0.1	−0.11	+0.15	+0.52	+0.5	+0.4
L	+27.0	+34.5	−20.63	−36.65	−106.60	−49.4	−21.9
E	0	0	0	+0.12	+0.11	+0.1	0
	″	″	″	″	″	″	″
$\delta l \div \sin N$	+0.108	+0.200	−0.217	−2.158	+1.747	+0.355	+0.101
$\delta v \div \sin (N-f)$	+0.002	+0.002	−0.001	−0.034	+0.025	+0.007	+0.003
$(N+f)$	+0.002	+0.002	−0.001	−0.020	+0.022	+0.006	+0.003
$\delta \log r \div \cos N$	−1	−2	0	−71	+18	+5	+1

IV.—$j = 0$; $j' = -1$.

i'	-3	-2	-1	0	1	2		4	5
i	$+4$	3	2	1	0	-1	—	-3	-4
h	-0.07	-0.17	-0.30	$+0.116$	$+9.171$	$+5.704$	$+2.37$	$+0.89$	$+0.32$
$aDah$		-0.50	-0.57	$+0.38$	$+1.560$	$+6.43$	$+5.02$	$+2.8$	
$De'h$	-8.1	-19.5	-35.5	$+13.8$	$+1083.89$	$+673.05$	$+277.7$	$+103.6$	$+36.7$
L		-1.5	-1.9	$+1.1$	-1.46	$+36.64$	$+18.6$	$+8.6$	
E	-8.1	-19.5	-35.5	$+13.8$	$+1083.86$	$+673.10$	$+277.8$	$+103.6$	$+36.7$
		″	″	″	″	″	″	″	
$\delta l \div \sin N$		-0.006	-0.011	$+0.012$	-0.086	-0.600	-0.134	-0.040	
	″								
$\delta v \div \sin (N - f)$	$+0.049$	$+0.156$	$+0.410$	-0.291	(-127.63)	$+22.056$	$+3.995$	$+0.955$	$+0.249$
$N - 2f$	$+0.001$	$+0.002$	$+0.004$	-0.003	-1.348	$+0.233$	$+0.042$	$+0.010$	$+0.003$
$N - 3f$	0	0	0	0	-0.015	$+0.003$			
$\delta \log r \div \cos N$	0	0	0	0	$+29$	-9	-2	-1	0
$(N - f)$	$+1$	$+2$	$+4$	-3	-1344	$+232$	$+42$	$+10$	$+3$
$(N - 2f)$	0	0	0	0	-17	$+3$	$+1$		

V.—$j = -2$; $j' = 1$.

i'	-2	-1	0	1	2	3
i	$+3$	2	1	0	-1	-2
$De'h$	$+0.7$	-0.04	$+0.07$	-0.26	-0.44	-0.39
	″	″	″	″	″	″
$\delta v \div \sin (N - f)$	-0.006	0.000	-0.002	$+0.031$	-0.014	-0.006

VI.—$j = 1$; $j' = -2$.

i'	-2	-1	0	1	2	3
i	$+3$	2	1	0	-1	-2
$De'h$				-0.11	-0.14	$+0.46$
	″	″	″	″	″	″
$\delta v \div \sin (N - f)$	0.000	0.000	0.000	$+0.013$	-0.005	$+0.007$

VII.—$j = -2$; $j' = 0$.

i'	-1	0	1	2	3	4	5
i	$+3$	2	1	0	-1	-2	-3
h	$+0.3$	-0.03	$+0.18$	$+0.66$	$+0.48$	$+0.27$	
$aDah$	$+0.4$	-0.05	$+0.34$	$+1.44$	$+1.51$	$+1.08$	
L	$+1.5$	-0.2	$+0.95$	$+2.22$	$+5.7$	$+3.2$	
	″	″	″	″	″	″	
$\delta l \div \sin N$	$+0.006$	-0.001	$+0.008$	$+0.065$	-0.129	-0.026	

VIII.—$j = -1$; $j' = 1$.

i'	−1	0	1	2	3	4	5
i	+3	2	1	0	−1	−2	−3
h	−0.02	0.0	−0.04	−0.50	−0.41	−0.23	−0.11
$aDah$	−0.04	−0.02	−0.11	−0.61	−0.88	−0.72	−0.45
$De'h$	−2.1	−0.49	−5.17	−59.02	−47.95	−26.96	−12.92
L	−0.1	0.0	−0.30	−0.96	−4.20	−2.41	−1.40
			″	″	″	″	″
$\delta l \div \sin N$	0	0	−0.003	−0.028	+0.095	+0.020	+0.007
	″	″					
$\delta v \div \sin (N - f)$	+0.016	+0.005	+0.092	+3.477	−2.178	−0.442	−0.129
$N - 2f$	0	0	+0.001	+0.037	−0.023	−0.005	−0.001
$\delta \log r \div \cos (N - f)$	0	0	+1	+37	−23	−5	−1

IX.—$j = 0$; $j' = -2$.

i'	−1	0	1	2	3	4	5
i	+3	2	1	0	−1	−2	−3
h	0	0	+0.004	+0.081	+0.083	+0.05	+0.02
$aDah$	0	0	+0.008	+0.022	+0.098	+0.11	+0.08
$De'h$	+0.08	+0.05	+1.00	+19.12	+19.63	+11.64	+5.71
L	0	0	+0.02	+0.05	+0.72	+0.45	+0.24
				″	″	″	″
$\delta l \div \sin (N - f)$	0	0	0	+0.001	−0.016	−0.004	−0.001
	″	″	″				
$\delta v \div \sin (N - f)$	−0.001	−0.001	−0.018	−1.126	+0.891	+0.191	+0.057
$N - 2f$				−0.012	+0.009	+0.002	+0.001
$\delta \log r \div \cos (N - f)$				−12	+9	+2	+1

X.—$j = 0$; $j' = 0$.

i'	−1	0	1	2	3	4
i	+3	2	1	0	−1	−2
$\frac{1}{4} Duh$	+0.30	+0.77	+1.68	+0.77	+0.30	+0.11
	″	″	″	″	″	″
$\delta\beta \div \sin (N - \lambda)$	+0.002	+0.008	+0.030	+0.045	−0.013	−0.002

XI.—$j = -2$; $j' = -1$.

i'	+2	3	4	5	6
i	+1	0	−1	−2	−3
$De'h$	+0 43	+2.34	+2.44	+1.72	+0.7
	″	″	″	″	″
$\delta v \div \sin (N - f)$	−0.007	−0.092	+0.180	+0 033	+0.008

XII.—$j = -1$; $j' = -2$.

i'	$+2$	3	4	5	6
i	$+1$	0	-1	-2	-3
$De'h$	-0.14	-1.75	-2.00	-1.5	-1.0
	$''$	$''$	$''$	$''$	$''$
$\delta v \div \sin(N-f)$	$+0.001$	$+0.068$	-0.148	-0.028	-0.016

XIII.—$j = 0$; $j' = -3$.

i'	$+2$	3	4	5	6
i	$+1$	0	-1	-2	-3
$De'h$	$+0.02$	$+0.28$	$+0.40$	$+0.30$	$+0.3$
	$''$	$''$	$''$	$''$	$''$
$\delta v \div \sin(N-f)$	0	-0.011	$+0.030$	$+0.006$	$+0.003$

ACTION OF JUPITER.

The direct action of Jupiter is so nearly insignificant that the details of the computation are omitted. The only terms in the longitude exceeding one hundredth of a second, and not sensibly confounded with the elliptic elements of Neptune, are

$$0''.278 \sin (\lambda' - \lambda)$$
$$+ 0 .019 \sin 2 (\lambda' - \lambda$$

ACTION OF VENUS, EARTH, AND MARS.

The only appreciable effect of the attraction of these planets is found in the relation between the radius vector and the mean motion. The coefficients of the perturbative function which correspond to the case when both i' and i are zero introduce changes as below into the secular variation of the longitude of the epoch. Those which correspond to the term in which $N = \lambda' - \omega'$ introduce constants as below into the logarithm of the radius vector. For the sake of completeness we include the similar perturbations produced by Jupiter, Saturn, and Uranus, as already computed:

	$\frac{d\varepsilon}{dt}$	$\delta \log r$
Action of Venus,	$+ 0''.0403$	-11
Earth,	$+ 0 .0444$	-12
Mars,	$+ 0 .0059$	-2
Jupiter,	$+ 15 .3571$	-4240
Saturn,	$+ 4 .8687$	-1344
Uranus,	$+ 1 .1261$	-311
Total,	21.4425	-5920

The principal term of $\frac{d\varepsilon}{dt}$, and, indeed, the entire portion not multiplied by the second power of the eccentricity, is

$$\frac{d\varepsilon}{dt} = mn' \left(b_{\frac{1}{2}}^{(0)} + a \frac{db_{\frac{1}{2}}^{(0)}}{da}\right);$$

while the principal term in $\delta \log r$ is

$$\delta \log r = - \tfrac{1}{2}\, mM \left(b_{\frac{1}{2}}^{(0)} + aDab_{\frac{1}{2}}^{(0)}\right).$$

The effect of these terms might, therefore, have been included in the mean distance as a single term, without appreciable error.

§ 14. *Perturbations of Neptune by Saturn through the Sun.*

These perturbations, it will be remembered, have been omitted in the preceding computations, from reasons already set forth. They have been computed by formulæ (16)—(19), and are as follows:

ACTION OF SATURN.

$\delta v =$	$\delta \log r =$
$-20''.536 \sin(\lambda' - \lambda)$	$+345 \cos(\lambda' - \lambda)$
$-\ 0.007 \sin(2\lambda' - 2\lambda - \omega' + \omega)$	
$+\ 0.530 \sin(-\lambda' + 2\lambda - \omega)$	$+\ 10 \cos(-\lambda' + 2\lambda - \omega)$
$-\ 0.059 \sin(\lambda - \omega')$	$-\ 2 \cos(\lambda - \omega')$
$-\ 0.340 \sin(2\lambda' - \lambda - \omega')$	$+\ 3 \cos(2\lambda' - \lambda - \omega')$
$+\ 0.022 \sin(-\lambda' + 3\lambda - 2\omega)$	
$-\ 0.007 \sin(\lambda' + \lambda - 2\omega)$	
$-\ 0.002 \sin(2\lambda - \omega - \omega')$	

§ 15. *Perturbations of the elements.*—Collecting and adding up the coefficients of all sines or cosines of the same angle in the perturbations, we find them as below. For λ and ω, their values, $l - \tau$ and $\pi - \tau$, are substituted. We shall first collect those terms which are developed as perturbations of the elements, namely, the secular variations, and all terms in the action of Uranus in which $i' = 2i$. We find them to be as follows:

$$
\begin{aligned}
\delta h = &+ 125''.67 \sin(2l' - l) \\
&- 0.42 \sin(2l' - l - 2\pi) \\
&- 0.36 \sin(2l' - l + \pi - \pi') \\
&+ 0.14 \sin(2l' - l + \pi' - \pi) \\
&- 30''.93 \sin(4l' - 2l - \pi) \\
&+ \ 8.03 \sin(4l' - 2l - \pi') \\
&- \ 0.03 \sin(4l' - 2l + \pi' - 2\pi) \\
&+ 2''.62 \sin(6l' - 3l - 2\pi) \\
&- 1.37 \sin(6l' - 3l - \pi' - \pi) \\
&+ 0.17 \sin(6l' - 3l - 2\pi') \\
&+ 0''.0132\, t
\end{aligned}
$$

$$
\begin{aligned}
\delta k = &+ 125''.67 \cos(2l' - l) \\
&+ 0.42 \cos(2l' - l - 2\pi) \\
&- 0.36 \cos(2l' - l - \pi' + \pi) \\
&+ 0.14 \cos(2l' - l + \pi' - \pi) \\
&- 30''.93 \cos(4l' - 2l - \pi) \\
&+ \ 8.03 \cos(4l' - 2l - \pi') \\
&- \ 0.03 \cos(4l' - 2l + \pi' - 2\pi) \\
&+ 2''.62 \cos(6l' - 3l - 2\pi) \\
&- 1.37 \cos(6l' - 3l - \pi' - \pi) \\
&+ 0.17 \cos(6l' - 3l - 2\pi') \\
&+ 0''.0031\, t
\end{aligned}
$$

$$
\begin{aligned}
\delta l = &+ 2163''.60 \sin(2l' - l - \pi) \\
&- \ 141.69 \sin(2l' - l - \pi') \\
&+ \ \ 0.56 \sin(2l' - l + \pi' - 2\pi) \\
&+ \ \ 0.21 \sin(2l' - l + \pi - 2\pi') \\
&+ \ \ 1.08 \sin(2l' - l + \pi - 2\tau) \\
&- \ \ 0.08 \sin(2l' - l + \pi' - 2\tau) \\
&- 71''.93 \sin(4l' - 2l - 2\pi) \\
&+ 38.09 \sin(4l' - 2l - \pi' - \pi) \\
&- \ 4.99 \sin(4l' - 2l - 2\pi') \\
&+ 4''.36 \sin(6l' - 3l - 3\pi) \\
&- 3.27 \sin(6l' - 3l - \pi' - 2\pi) \\
&+ 0.85 \sin(6l' - 3l - 2\pi' - \pi) \\
&- 0.08 \sin(6l' - 3l - 3\pi') \\
&+ 21''.4425\, t
\end{aligned}
$$

$$
\begin{aligned}
\delta \log a = &- 1232 \cos(2l' - l - \pi) \\
&+ \ \ 92 \cos(2l' - l - \pi') \\
&+ \ \ 85 \cos(4l' - 2l - 2\pi) \\
&- \ \ 44 \cos(4l' - 2l - \pi' - \pi) \\
&+ \ \ \ 6 \cos(4l' - 2l - 2\pi')
\end{aligned}
$$

$$\begin{aligned}\delta p = &-1''.11 \sin(2l'-l-\pi+\tau)\\ &-0.72 \sin(2l'-l-\pi-\tau)\\ &+0.16 \sin(2l'-l-\pi'+\tau)\\ &+0.15 \sin(2l'-l-\pi'-\tau)\\[6pt] &-2''.98 \sin(4l'-2l-\tau)\\ &+0''.0110\,t\end{aligned} \qquad \begin{aligned}\delta q = &-1''.11 \cos(2l'-l-\pi+\tau)\\ &+0.72 \cos(2l'-l-\pi-\tau)\\ &+0.16 \cos(2l'-l-\pi'+\tau)\\ &-0.15 \cos(2l'-l-\pi'-\tau)\\[6pt] &-2''.98 \cos(4l'-2l-\tau)\\ &+0''.0001\,t\end{aligned}$$

§ 16. *Perturbations of the co-ordinates—Comparison with* PEIRCE *and* KOWALSKI. —The first column of the following tables gives the coefficients according to Peirce (Proceedings of the American Academy, Vol. 1, pp. 287–291); and the second, the values according to Kowalski (Recherches sur les mouvements de Neptune, pp. 14–16). In the case of Uranus, Peirce's coefficients have been increased by $\frac{1}{6}+\frac{1}{50}$, to reduce his mass of Uranus to the adopted one. The coefficients enclosed in parentheses are not comparable, as they include the effect of terms now developed as perturbations of the elements, and therefore omitted from the perturbations of the co-ordinates. The perturbations of the radius vector have been reduced to logarithms by multiplying by $\frac{0.4342}{30}$.

I.—ACTION OF URANUS.

		$\delta v =$			$\delta \log r =$
P.	K.	N.	P.	K.	N.
$(-206''.91)$	$(-244''.40)$	$+3''.002 \sin (l'-l)$	(-2284)	(-2289)	$+314 \cos (l'-l)$
$+10.24$	$+10.02$	$+9.994 \sin 2(l'-l)$	$+167$	$+163$	$+162 \cos 2(l'-l)$
$+2.01$	$+2.02$	$+1.960 \sin 3(l'-l)$	$+40$	$+69$	$+38 \cos 3(l'-l)$
$+0.64$	$+0.62$	$+0.610 \sin 4(l'-l)$	$+14$	$+38$	$+13 \cos 4(l'-l)$
$+0.25$	$+0.27$	$+0.237 \sin 5(l'-l)$	$+5$	$+23$	$+5 \cos 5(l'-l)$
$+0.11$	$+0\ 35$	$+0.104 \sin 6(l'-l)$	$+2$	$+11$	$+1 \cos 6(l'-l)$
$+0.05$	$+0.27$	$+0.041 \sin 7(l'-l)$			
$+0.02$		$+0.017 \sin 8(l'-l)$			
$+0.01$		$+0.007 \sin 9(l'-l)$			
		$+0''.002 \sin(-4l'+4l-\pi'+\pi)$			
		$+0.016 \sin(-3l'+3l-\pi'+\pi)$			
(-0.11)	(-0.73)	$-0.103 \sin(-2l'+2l-\pi'+\pi)$			
(-16.29)	(-16.79)	$-0.048 \sin(-l'+l-\pi'+\pi)$			
$(+0.66)$	$(+0.71)$	$+0.045 \sin(l'-l-\pi'+\pi)$			
		$-0.011 \sin(2l'-2l-\pi'+\pi)$			
		$+0.003 \sin(3l'-3l-\pi'+\pi)$			
		$+0.003 \sin(4l'+4l-\pi'+\pi)$			
		$+0.002 \sin(5l'+5l-\pi'+\pi)$			
-0.01		$-0''.009 \sin(-5l'+6l-\pi)$			
-0.01		$-0.014 \sin(-4l'+5l-\pi)$			
-0.02		$-0.024 \sin(-3l'+4l-\pi)$			
-0.04	-0.08	$-0.033 \sin(-2l'+3l-\pi)$			
$+0.19$	$+0.19$	$+0.183 \sin(-l'+2l-\pi)$	$+2$	$+2$	$+3 \cos(-l'+2l-\pi)$
$+0.27$	-1.31	$+0.274 \sin(l-\pi)$	-5	$+11$	$-5 \cos(l-\pi)$
		$-0.238 \sin(l'-\pi)$			$-10 \cos(l'-\pi)$
(1979.72)	(1955.50)	$+4.365 \sin(2l'-l-\pi)$	(-1141)	(-1127)	$+43 \cos(2l'-l-\pi)$
$(+69.86)$	$(+68.73)$	$+9.563 \sin(3l'-2l-\pi)$	$(+693)$	$(+663)$	$+58 \cos(3l'-2l-\pi)$
-1.78	-1.78	$-1.721 \sin(4l'-3l-\pi)$	-28	-27	$-27 \cos(4l'-3l-\pi)$
-0.33	-0.59	$-0.375 \sin(5l'-4l-\pi)$	-7	-5	$-7 \cos(5l'-4l-\pi)$
-0.12	-0.29	$-0.134 \sin(6l'-5l-\pi)$	-3	-1	$-2 \cos(6l'-5l-\pi)$
-0.06		$-0.057 \sin(7l'-6l-\pi)$	-2		$-2 \cos(7l'-6l-\pi)$
-0.04		$-0.022 \sin(8l'-7l-\pi)$			$-2 \cos(8l'-7l-\pi)$
$-0{\cdot}01$		$-0.009 \sin(9l'-8l-\pi)$			

ACTION OF URANUS (Continued).

		$\delta v =$			$\delta \log r =$
P.	*K.*	*N.*	*P.*	*K.*	*N.*
		$+0''.001 \sin(-5l'+6l-\pi')$			
		$+0\ .002 \sin(-4l'+5l-\pi')$			
		$-0\ .002 \sin(-3l'+4l-\pi')$			$+1 \cos(-3l'+4l-\pi)$
(-0.01)		$-0\ .015 \sin(-2l'+3l-\pi')$			$-1 \cos(-2l'+3l-\pi)$
(-0.11)		$-0\ .109 \sin(-l'+2l-\pi')$	$+2$		$+2 \cos(-l'+2l-\pi)$
$(+2.33)$	$(+2.65)$	$-0\ .177 \sin(l-\pi')$	(-21)	(-24)	$+2 \cos(l-\pi)$
		$+0\ .209 \sin(l'-\pi')$			$+5 \cos(l'-\pi)$
(-124.83)	(-132.51)	$-0\ .466 \sin(2l'-l-\pi')$	$(+95)$	$(+97)$	$-13 \cos(2l'-l-\pi)$
(-17.45)	(-18.37)	$-2\ .477 \sin(3l'-2l-\pi')$	(-174)	(-184)	$-15 \cos(3l'-2l-\pi)$
$+0.46$	$+0.53$	$+0\ .452 \sin(4l'-3l-\pi')$	$+7$	$+6$	$+9 \cos(4l'-3l-\pi)$
$+0.11$	$+0.07$	$+0\ .101 \sin(5l'-4l-\pi')$	$+2$	$+1$	$+1 \cos(5l'-4l-\pi)$
$+0.04$	-0.23	$+0\ .027 \sin(6l'-5l-\pi')$		3	$+1 \cos(6l'-5l-\pi)$
$+0.01$		$+0\ .014 \sin(7l'-6l-\pi')$			
		$+0\ .010 \sin(8l'-7l-\pi')$			
		$+0\ .006 \sin(9l'-8l-\pi')$			
		$+0''.002 \sin(3l'-2l+\omega'-2\pi)$			
		$-0\ .016 \sin(4l'-3l+\omega'-2\pi)$			
		$+0\ .002 \sin(5l'-4l+\omega'-2\pi)$			

(The terms in which the constant of the argument is $\pi - 2\pi$, $\pi - 2\tau$, and $\pi - 2\tau'$ are yet smaller, and are neglected.)

P.	*K.*	*N.*
		$+0''.017 \sin(-l'+3l-2\pi)$
		$-0\ .001 \sin(2l-2\pi)$
		$-0\ .007 \sin(l'+l-2\pi)$
		$-0\ .009 \sin(2l'-2\pi)$
	$(+0.75)$	$-0\ .151 \sin(3l'-l-2\pi)$
(-65.45)	(-64.61)	$-0\ .587 \sin(4l'-2l-2\pi)$
(-5.10)	(-5.62)	$-1\ .310 \sin(5l'-3l-2\pi)$
		$+0\ .258 \sin(6l'-4l-2\pi)$
		$+0\ .061 \sin(7l'-5l-2\pi)$
		$+0\ .027 \sin(8l'-6l-2\pi)$
		$+0\ .010 \sin(9l'-7l-2\pi)$
		$+0\ .004 \sin(10l'-8l-2\pi)$
		$+0''.005 \sin(l'+l-\pi'-\pi)$
		$+0\ .006 \sin(2l'-\pi'-\pi)$
$(+16.08)$	$(+17.01)$	$+0\ .098 \sin(3l'-l-\pi'-\pi)$
$(+33.73)$	$(+36.67)$	$+0\ .366 \sin(4l'-2l-\pi'-\pi)$
$(+2.56)$	$(+3.35)$	$+0\ .688 \sin(5l'-3l-\pi'-\pi)$
		$-0\ .136 \sin(6l'-4l-\pi'-\pi)$
		$-0\ .032 \sin(7l'-5l-\pi'-\pi)$
		$-0\ .019 \sin(8l'-6l-\pi'-\pi)$
		$-0\ .010 \sin(9l'-7l-\pi'-\pi)$
		$-0\ .005 \sin(10l'-8l-\pi'-\pi)$
		$-0''.003 \sin(2l-2\pi')$
		$-0\ .011 \sin(l'+l-2\pi')$
(-1.04)	(-1.15)	$+0\ .005 \sin(3l'-l-2\pi')$
(-4.29)	(-4.79)	$-0\ .046 \sin(4l'-2l-2\pi')$
(-0.33)	(-0.21)	$-0\ .090 \sin(5l'-3l-2\pi')$
		$+0\ .020 \sin(6l'-4l-2\pi')$
		$+0\ .005 \sin(7l'-5l-2\pi')$
		$+0\ .002 \sin(8l'-6l-2\pi')$
		$-0''.003 \sin(l'+l-2\tau)$
		$-0\ .016 \sin(3l'-l-2\tau)$
		$-0\ .022 \sin(4l'-2l-2\tau)$
		$-0\ .041 \sin(5l'-3l-2\tau)$
		$+0\ .006 \sin(6l'-4l-2\tau)$

Latitude.

$\delta\beta =$
N.
$-0''.004 \sin(-4l'+5l-\tau)$
$-0\ .008 \sin(-3l'+4l-\tau)$
$-0\ .023 \sin(-2l'+3l-\tau)$
$-0\ .056 \sin(-l'+2l-\tau)$
$+0\ .040 \sin(l-\tau)$
$+0\ .106 \sin(l'-\tau)$
$+0\ .320 \sin(2l'-l-\tau)$
$+0\ .060 \sin(3l'-2l-\tau)$
$-0\ .063 \sin(4l'-3l-\tau)$
$-0\ .016 \sin(5l'-4l-\tau)$
$-0\ .005 \sin(6l'-5l-\tau)$
$-0\ .004 \sin(7l'-6l-\tau)$

II.—ACTION OF SATURN.

$\delta v =$ P.	K.	N.	$\delta \log r =$ P.	K.	N.
$-18''.60$	$-18''.12$	$-18''.552 \sin (l' - l)$	$+398$	$+393$	$+397 \cos (l' - l)$
$+0.15$	$+0.15$	$+0.141 \sin 2 (l' - l)$	$+4$	0	$+4$
$+0.02$	$+0.03$	$+0.012 \sin 3 (l' - l)$			
	$+0.06$	$+0.000 \sin 4 (l' - l)$			
		$+0''.002 \sin (-l' + l - \pi' + \pi)$			
		$-0.006 \sin (2l' - 2l - \pi' + \pi)$			
$+0.54$	$+0.53$	$+0''.524 \sin (-l' + 2l - \pi)$	$+12$	$+11$	$+9 \cos (-l' + 2l - \pi)$
$+0.01$	-0.16	$+0.008 \sin (l - \pi)$		-2	$+2 \cos (l - \pi)$
		$+1.319 \sin (l' - \pi)$			$-34 \cos (l' - \pi)$
-0.28	$+1.09$	$-0.280 \sin (2l' - l - \pi)$	-6	-20	$-7 \cos (2l' - l - \pi)$
-0.02	-0.17	$-0.023 \sin (3l' - 2l - \pi)$			$-1 \cos (3l' - 2l - \pi)$
		$-0.004 \sin (4l' - 3l - \pi)$			
-0.08	-0.09	$-0''.080 \sin (l - \pi')$	-3	-3	$-1 \cos (l - \pi')$
		$+0.136 \sin (l' - \pi')$			
-0.22	$+3.85$	$-0.228 \sin (2l' - l - \pi')$	$+3$	$+47$	$+5 \cos (2l' - l - \pi')$
$+0.01$	$+0.04$	$+0.008 \sin (3l' - 2l - \pi')$			
		$+0.001 \sin (4l' - 3l - \pi')$			
		$+0''.022 \sin (-l' + 3l - 2\pi)$			
	$+0.10$	$-0.008 \sin (l' + l - 2\pi)$			
		$+0.004 \sin (2l' - 2\pi)$			
	$+0.13$	$+0.037 \sin (3l' - l - 2\pi)$			
		$-0''.002 \sin (2l - \pi' - \pi)$			
		$-0.002 \sin (l' + l - \pi' - \pi)$			
		$+0.020 \sin (2l' - \pi' - \pi)$			
	$+0.10$	$-0.029 \sin (3l' - l - \pi' - \pi)$			
		$+0''.005 \sin (2l' - 2\pi')$			
	-0.75	$+0.006 \sin (3l' - l - 2\pi')$			

$$
\begin{aligned}
\delta\beta = &+ 0''.309 \sin (l - \tau) \\
&+ 0.045 \sin (l' - \tau) \\
&- 0.005 \sin (2l' - l - \tau)
\end{aligned}
$$

III.—ACTION OF JUPITER.

$\delta v =$ P.	K.	N.	$\delta \log r =$ P.	K.	N.
$-34''.09$	$-32''.67$	$-34''.121 \sin (l' - l)$	$+719$	$+683$	$+701 \cos (l' - l)$
$+0.02$	$+0.03$	$+0.019 \sin 2 (l' - l)$		0	$+1 \cos 2 (l' - l)$
	-0.14	$+0.003 \sin 3 (l' - l)$			
	$+0.11$	$-0.009 \sin (2l' - 2l - \pi' + \pi)$			
$+0.82$	$+0.84$	$+0''.801 \sin (-l' + 2l - \pi)$	$+17$	$+17$	$+18 \cos (-l' + 2l + \pi)$
	-0.07	$+0.003 \sin (l - \pi)$			$+51 \cos (-l' - \pi)$
		$+2.358 \sin (l' - \pi)$			
-0.01	$+0.19$	$-0.010 \sin (2l' - l - \pi)$			
-0.14	-0.15	$-0''.143 \sin (l - \pi')$	-6	-27	$-2 \cos (l - \pi')$
		$+0.117 \sin (l' - \pi')$			$-2 \cos (l' - \pi')$
-0.42	-0.48	$-0.432 \sin (2l' - l - \pi')$	$+6$	$+135$	$+7 \cos (2l' - l - \pi')$

ACTION OF JUPITER (Continued).

	$\delta v =$			$\delta \log r =$	
P.	*K.*	*N.*	*P.*	*K.*	*N.*
		$+ 0''.030 \sin(- l' + 3l - 2\pi)$			
		$- 0\ .011 \sin(l' + l - 2\pi)$			
		$+ 0\ .004 \sin(2l' - 2\pi)$			
	$+ 0.10$	$- 0''.005 \sin(2l - \pi' - \pi)$			
		$+ 0\ .028 \sin(2l' - \pi' - \pi)$			

$\delta\beta =$

$+ 0''.564 \sin(l - \tau)$
$+ 0\ .039 \sin(l' - \tau)$

By comparing the different authorities for the coefficients, it will be seen that while our present results agree very well with those of Professor Peirce, the agreement with Professor Kowalski is in many cases very far from being satisfactory. It will be observed that the latter differ most in the case of those terms whose coefficients depend on the action of the disturbing planets on the Sun, and we have also seen that these terms are ordinarily developed as small differences of very large quantities. They are, therefore, the terms into which errors would most easily creep.

The terms enclosed in parentheses are not of great importance, because they are for a long period sensibly confounded with the elliptic elements. Notwithstanding that one of these terms amounts to more than half a degree, and others to several minutes, the effect of the whole of them could scarcely be discovered from all the observations hitherto made on Neptune.

§ 17. For the purpose of tabulating and computing an ephemeris, it is expedient to change the form of the perturbations by Uranus. Consider any two terms in which the coefficients of l are equal, but of opposite signs:

$$\delta v = p_1 \sin\{sl' - iA - \omega\} + p_2 \sin\{sl' + iA - \omega\}$$

where

$$A = l' - l$$

The terms may then be put in the form

$$\{(p_2 - p_1) \sin\omega \sin iA + (p_2 + p_1) \cos\omega \cos iA\} \sin sl'$$
$$\{(p_2 - p_1) \cos\omega \sin iA - (p_2 + p_1) \sin\omega \cos iA\} \cos sl'$$

So that we may put

$$\delta v \quad = \delta v_o \quad + P_{s.1} \sin l' + P_{c.1} \cos l' + P_{s.2} \sin 2l' + P_{c.2} \cos 2l'$$
$$\delta \log r = \delta \log r_o + R_{s.1} \sin l' + P_{c.1} \cos l'$$

where δv, P, and R are functions only of A, and may be tabulated as such.

§ 18. For Jupiter and Saturn, if we neglect those terms of which the coefficients are less than 0''.03, it will be more convenient to tabulate the perturbations directly. This course we shall adopt, except with reference to those perturbations which depend on the mean longitude of Neptune alone, and do not contain the mean longitude of the disturbing planets. These have been omitted by both

Peirce and Kowalski, as may be seen by reference to the preceding values of their coefficients. They are, in fact, very nearly confounded with the elliptic motion of the planet, but not exactly. We shall, at present, retain only the small residuals, after subducting those portions which are sensibly elliptic. The entire terms are as follows:

1. In the longitude.

Action of Uranus,	$+0''.385 \sin l$	$-0''.092 \cos l$	$-0''.014 \sin 2l$	$-0''.002 \cos 2l$	
Saturn,	$+0\ .099 \sin l$	$-1\ .412 \cos l$	$-0\ .018 \sin 2l$	$-0\ .020 \cos 2l$	
Jupiter,	$+2\ .393 \sin l$	$-0\ .567 \cos l$	$+0\ .018 \sin 2l$	$-0\ .029 \cos 2l$	
Total,	$+2\ .877 \sin l$	$-2\ .071 \cos l$	$-0\ .014 \sin 2l$	$-0\ .051 \cos 2l$	(a)

2. In the logarithm of radius vector.

Action of Uranus,	$+1 \sin l$	$+14 \cos l$	
Saturn,	$-34 \sin l$		
Jupiter,	$-11 \sin l$	$-51 \cos l$	
Total,	$-44 \sin l$	$-37 \cos l$	(b)

Changes in the functions $e \sin \pi$ and $e \cos \pi$, represented by δh and δk, will produce the following changes in the longitude and log r,

$$\delta v = 2\,\delta k \sin l - 2\,\delta h \cos l + \tfrac{5}{2}(k\delta k - h\delta h) \sin 2l - \tfrac{5}{2}(k\delta h + h\delta k) \cos 2l$$
$$\delta \log r = -M\delta h \sin l - M\delta k \cos l.$$

Taking the elliptic terms to be subducted so that the coefficients of sin l and cos l shall vanish, we must put

$$\delta h = +1''.036\,; \quad \delta k = +1''.438,$$

which will produce the inequalities

$$\delta v = +2''.877 \sin l - 2''.071 \cos l + 0''.007 \sin 2l - 0''.037 \cos 2l$$
$$\delta \log r = -21 \sin l - 30 \cos l.$$

Subtracting these elliptic inequalities from (a) and (b), we have for the residuals

$$\delta v = -0''.021 \sin 2l - 0''.014 \cos 2l$$
$$\delta \log r = -23 \sin l - 7 \cos l.$$

So that the constants of P_s, etc. are

$$\text{Constant of } P_{s.1} = 0$$
$$P_{c.1} = 0$$
$$P_{s.2} = -0''.021$$
$$P_{c.2} = -0\ .014$$
$$R_{s.1} = -23$$
$$R_{c.1} = -7$$

The constant terms in the coefficients $B_{s,1}$ and $B_{c,1}$, which give the perturbations of the latitude, may be omitted without any error amounting to one hundredth of a second.

§ 19. The form of the preceding perturbations being different from that of the perturbations computed by Professor Peirce, the elliptic elements are next provisionally altered, so that the provisional theory shall be substantially identical with that already adopted. Small corrections have also been applied to the constants which determine the plane of the orbit.

The provisional elements finally adopted for correction are as follows:

$$\begin{aligned}
\varepsilon &= 335^\circ\ 5'\ 25''.97 \\
n &= 7864.421 \\
h &= +1192.93 \\
k &= +1279.36 \\
p &= +4910.17 \\
q &= -4137.46
\end{aligned}$$

Epoch, 1850, Jan. 0, Greenwich mean noon. Unit of time, 365.25 days.

$e =$	0.00848055
e (in seconds)	1749''.24
i	1° 47′ 1''.95
π	42 59 52 .0
Ω	130 7 6 .7
$\log a$	1.4787523

The perturbations of the preceding elements are expressed in the following form:

Put $M = 2l' - l$

$T =$ Number of centuries after 1850, Jan. 0.

Then, $M = 281^\circ\ 43'\ 48'' + 8^\circ\ 26'\ 10''.7\ T;$

and

$$\begin{aligned}
\delta h = {} & 125''.42 \sin (M - 0^\circ\ 16'.3) \\
& + 36.08 \sin (2M + 1^\circ\ 50') \\
& + 3.58 \sin (3M + 3^\circ\ 42') \\
& + 1''.32\ T + \text{constant.}
\end{aligned}
\qquad
\begin{aligned}
\delta k = {} & 126''.17 \cos (M - 0^\circ\ 6'.2) \\
& + 36.08 \cos (2M + 1^\circ\ 50') \\
& + 3.58 \cos (3M + 3^\circ\ 42') \\
& + 0''.31\ T + \text{constant.}
\end{aligned}$$

$$\begin{aligned}
\delta l = {} & 2247''.52 \sin (M - 170^\circ\ 32'\ 23'') \\
& + 98.57 \sin (2M + 183^\circ\ 24'.1) \\
& + 6.81 \sin (3M + 186^\circ\ 14') \\
& + 2144''.26\ T + \text{const.} + \text{const.} \times T.
\end{aligned}
\qquad
\begin{aligned}
\delta \log a = {} & 1286 \cos (M + 9^\circ\ 8') \\
& + 115 \cos (2M + 4^\circ\ 0') \\
& + \text{constant.}
\end{aligned}$$

$$\begin{aligned}
\delta p = {} & 1''.86 \sin M \\
& + 2.98 \sin (2M - 155^\circ\ 38') \\
& + 1''.10\ T + \text{constant.}
\end{aligned}
\qquad
\begin{aligned}
\delta q = {} & 0''.87 \cos (M - 61^\circ\ 0') \\
& + 2.98 \cos (2M - 155^\circ\ 38') \\
& + 0''.01\ T + \text{constant.}
\end{aligned}$$

The constants being so taken that the perturbations, and also the differential coefficient of δl, shall all vanish at the epoch 1850.0. These perturbations are given for the beginning of every tenth year, from 1600 to 2000, in the following table:

Secular and Long-Period Perturbations of the Elements of Neptune from 1600 to 2000.

Date	δl	$\delta \log a$	δh	δk	δp	δq	$\delta p'$	$\delta q'$
	″		″	″	″	″	″	″
1600	— 149.21	— 473	+ 22.33	— 45.28	— 4.68	+ 0.79	+ 8.58	— 116.52
10	137.63	455	21.18	43.85	4.48	0.79	8.29	111.82
20	126.50	437	20.05	42.39	4.29	0.78	7.99	107.12
30	115.83	418	18.94	40.90	4.10	0.77	7.69	102.42
40	105.62	400	17.85	39.37	3.90	0.76	7.39	97.73
50	— 95.88	— 381	+ 16.77	— 37.81	— 3.71	+ 0.75	+ 7.09	— 93.04
60	86.60	362	15.71	36.22	3.52	0.73	6.78	88.36
70	77.78	344	14.67	34.60	3.32	0.71	6.47	83.68
80	69.42	325	13.65	32.94	3.13	0.69	6.15	79.00
90	61.52	307	12.66	31.25	2.94	0.67	5.83	74.33
1700	— 54.09	— 288	+ 11.69	— 29.53	— 2.75	+ 0.64	+ 5.50	— 69.66
10	47.13	269	10.74	27.77	2.56	0.61	5.17	64.99
20	40.65	250	9.81	25.99	2.37	0.58	4.84	60.32
30	34.65	231	8.90	24.17	2.18	0.55	4.49	55.66
40	29.13	212	8.01	22.33	2.00	0.52	4.14	51.01
50	— 24.09	— 193	+ 7.14	— 20.45	— 1.81	+ 0.48	+ 3.79	— 46.36
60	19.52	174	6.30	18.54	1.62	0.44	3.43	41.71
70	15.43	154	5.48	16.60	1.44	0.40	3.08	37.06
80	11.81	135	4.69	14.63	1.25	0.35	2.71	32.42
90	8.68	115	3.93	12.62	1.07	0.30	2.34	27.79
1800	— 6.03	— 96	+ 3.20	— 10.59	— 0.89	+ 0.25	+ 1.96	— 23.15
10	3.86	77	2.50	8.53	0.71	0.20	1.58	18.52
20	2.17	57	1.83	6.44	0.53	0.15	1.20	13.88
30	0.96	38	1.19	4.32	0.35	0.10	0.81	9.25
40	— 0.24	— 19	+ 0.58	— 2.17	— 0.17	+ 0.05	+ 0.41	— 4.63
50	0.00	0	0.00	0.00	0.00	0.00	0.00	0.00
60	— 0.24	+ 19	— 0.54	+ 2.20	+ 0.17	0.06	— 0.41	+ 4.62
70	0.96	38	1.04	4.42	0.34	0.11	0.82	9.25
80	2.17	57	1.51	6.68	0.51	0.17	1.24	13.87
90	3.86	77	1.93	8.96	0.68	0.23	1.67	18.48
1900	— 6.03	+ 96	— 2.32	+ 11.26	+ 0.85	— 0.29	— 2.10	+ 23.09
10	8.68	115	2.67	13.59	1.01	0.35	2.54	27.70
20	11.81	133	2.98	15.94	1.17	0.42	2.98	32.30
30	15.44	152	3.25	18.31	1.33	0.49	3.43	36.90
40	19.53	171	3.47	20.70	1.48	0.56	3.89	41.49
50	— 24.10	+ 190	— 3.65	+ 23.11	+ 1.64	— 0.63	— 4.35	+ 46.09
60	29.14	209	3.79	25.54	1.80	0.70	4.81	50.69
70	34.66	227	3.89	27.99	1.95	0.77	5.28	55.28
80	40.66	246	3.94	30.46	2.11	0.84	5.75	59.88
90	47.13	265	3.94	32.95	2.26	0.91	6.23	64.47
2000	— 54.09	+ 284	— 3.89	+ 35.46	+ 2.42	— 0.98	— 6.71	+ 69.06

δp and δq refer to the fixed ecliptic of 1850.0, $\delta p'$ and $\delta q'$ to the movable ecliptic of the date, the motion being that adopted in Hansen's "Tables du Soleil," and concluded from the secular diminution of the obliquity there given.

The corrections to the true longitude, latitude, and radius vector derived from the pure elliptic elements require corrections for these perturbations as follows:

$$\delta v = \frac{dv}{dl}\,\delta l + \frac{dv}{dh}\,\delta h + \frac{dv}{dk}\,\delta k,$$

$$\delta \log r = \delta \log a + \frac{d \log r}{dh}\,\delta h + \frac{d \log r}{dk}\,\delta k,$$

$$\delta \beta = \frac{d\beta}{dp}\,\delta p + \frac{d\beta}{dq}\,\delta q.$$

For the period during which Neptune has been observed, we have, to a sufficient degree of approximation,

$$\frac{dv}{dl} = 1,$$

$$\frac{dv}{dh} = -2 \cos l, \qquad \frac{dv}{dk} = 2 \sin l;$$

$$\frac{d \log r}{dh} = -M \sin l, \qquad \frac{d \log r}{dk} = -M \cos l;$$

$$\frac{d\beta}{dp} = -\cos v, \qquad \frac{d\beta}{dq} = \sin v.$$

The values of $P_{s.1}$, $P_{c.1}$, etc., derived from the perturbations by Uranus, are, putting $A =$ mean longitude of Uranus, minus that of Neptune,

$$\begin{array}{rll}
P_{s.1} = & -0''.683 \sin A & -\ 5''.000 \cos A \\
& -0\ .400 \sin 2A & -11\ .410 \cos 2A \\
& +0\ .044 \sin 3A & +\ 2\ .031 \cos 3A \\
& +0\ .006 \sin 4A & +\ 0\ .462 \cos 4A \\
& +0\ .009 \sin 5A & +\ 0\ .165 \cos 5A \\
& +0\ .001 \sin 6A & +\ 0\ .076 \cos 6A \\
& -0\ .003 \sin 7A & +\ 0\ .035 \cos 7A \\
& -0\ .002 \sin 8A & +\ 0\ .017 \cos 8A \\
& -0\ .002 \sin 9A & +\ 0\ .008 \cos 9A \\
& -0\ .001 \sin 10A & +\ 0\ .004 \cos 10A
\end{array}$$

$$\begin{array}{rll}
P_{c.1} = & +\ 4''.208 \sin A & -0''.559 \cos A \\
& +10\ .892 \sin 2A & -0\ .330 \cos 2A \\
& -\ 1\ .989 \sin 3A & +0\ .078 \cos 3A \\
& -\ 0\ .418 \sin 4A & +0\ .018 \cos 4A \\
& -\ 0\ .135 \sin 5A & +0\ .013 \cos 5A \\
& -\ 0\ .056 \sin 6A & +0\ .003 \cos 6A \\
& -\ 0\ .023 \sin 7A & \\
& -\ 0\ .009 \sin 8A & \\
& -\ 0\ .004 \sin 9A & \\
& -\ 0\ .002 \sin 10A &
\end{array}$$

$$\begin{array}{rll}
P_{s.2} = & -0''.021 & \\
& -0\ .254 \cos A & -0''.038 \sin A \\
& -0\ .867 \cos 2A & -0\ .035 \sin 2A \\
& -1\ .821 \cos 3A & -0\ .147 \sin 3A \\
& +0\ .355 \cos 4A & +0\ .023 \sin 4A \\
& +0\ .083 \cos 5A & +0\ .006 \sin 5A \\
& +0\ .039 \cos 6A & \\
& +0\ .018 \cos 7A & \\
& +0\ .008 \cos 8A & \\
& +0\ .004 \cos 9A &
\end{array}$$

$$\begin{array}{rll}
P_{c.2} = & -0''.014 & \\
& -0\ .022 \cos A & +0''.228 \sin A \\
& -0\ .027 \cos 2A & +0\ .863 \sin 2A \\
& -0\ .135 \cos 3A & +1\ .849 \sin 3A \\
& +0\ .025 \cos 4A & -0\ .355 \sin 4A \\
& +0\ .008 \cos 5A & -0\ 085 \sin 5A \\
& +0\ .002 \cos 6A & -0\ .041 \sin 6A \\
& & -0\ .018 \sin 7A \\
& & -0\ .008 \sin 8A \\
& & -0\ .004 \sin 9A
\end{array}$$

$$\begin{array}{rll}
R_{s.1} = & -23 & \\
& & -58 \sin A \\
& +\ 4 \cos 2A & -66 \sin 2A \\
& & +34 \sin 3A \\
& & +\ 7 \sin 4A \\
& & +\ 3 \sin 5A \\
& & +\ 2 \sin 6A
\end{array}$$

$$\begin{array}{rll}
R_{c.1} = & -7 & \\
& -46 \cos A & \\
& -70 \cos 2A & \\
& +32 \cos 3A & -1 \sin 3A \\
& +\ 9 \cos 4A & +2 \sin 4A \\
& +\ 3 \cos 5A & \\
& +\ 2 \cos 6A &
\end{array}$$

$$\begin{array}{rll}
B_{s.1} = & +0''.328 \cos A & +0''.116 \sin A \\
& +0\ .005 \cos 2A & +0\ .048 \sin 2A \\
& -0\ .078 \cos 3A & -0\ .017 \sin 3A \\
& -0\ .022 \cos 4A & -0\ .003 \sin 4A \\
& -0\ .009 \cos 5A & \\
& -0\ .006 \cos 6A &
\end{array}$$

$$\begin{array}{rll}
B_{c.1} = & +0''.148 \cos A & -0''.256 \sin A \\
& +0\ .002 \cos 2A & -0\ .105 \sin 2A \\
& -0\ .035 \cos 3A & +0\ .036 \sin 3A \\
& -0\ .009 \cos 4A & +0\ .008 \sin 4A \\
& -0\ .004 \cos 5A &
\end{array}$$

The other terms in the longitude, logarithm of r, and latitude, representing the mean longitude of the planet by the initial letter of its name, are:

$$
\begin{array}{llll}
\delta v_0 = -2''.949 \sin A & -0''.002 \cos A & \delta r_0 = \ \ 314 \cos A & \\
\quad\ -9.942 \sin 2A & -0.094 \cos 2A & \quad\ +162 \cos 2A & \\
\quad\ -1.967 \sin 3A & +0.016 \cos 3A & \quad\ +\ 38 \cos 3A & \\
\quad\ -0.610 \sin 4A & +0.004 \cos 4A & \quad\ +\ 13 \cos 4A & \\
\quad\ -0.237 \sin 5A & & \quad\ +\ \ 5 \cos 5A & \\
\quad\ -0.104 \sin 6A & & \quad\ +\ \ 2 \cos 6A & \\
\quad\ -0.041 \sin 7A & & & \\
\quad\ -0.017 \sin 8A & & & \\
\quad\ -0.007 \sin 9A & & & \\
& & & \\
\quad\ +18''.552 \sin (S-N) & & \quad\ +397 \cos (S-N) & \\
\quad\ -\ 0.137 \sin 2(S-N) & & \quad\ +\ \ 4 \cos 2(S-N) & \\
\quad\ -\ 0.012 \sin 3(S-N) & & & \\
& -0''.524 \cos (2S-N) & \quad\ +\ 10 \sin (2S-N) & +\ \ 1 \cos (2S-N) \\
\quad\ -\ 0.058 \sin S & +0.047 \cos S & \quad\ +\ \ 4 \sin (S-2N) & +\ \ 4 \cos (S-2N) \\
\quad\ +\ 0.166 \sin (S-2N) & -0.436 \cos (S-2N) & & +701 \cos (J-N) \\
\quad\ +34.121 \sin (J-N) & & \quad\ +\ \ 4 \sin (2J-N) & +\ 18 \cos (2J-N) \\
\quad\ -\ 0.011 \sin 2(J-N) & & \quad\ -\ \ 5 \sin (J-2N) & +\ \ 4 \cos (J-2N) \\
\quad\ +\ 0.783 \sin (2J-N) & -0.164 \cos (2J-N) & & \\
\quad\ -\ 0.101 \sin J & +0.097 \cos J & & \\
\quad\ +\ 0.326 \sin (J-2N) & +0.297 \cos (J-2N) & &
\end{array}
$$

$$\delta\beta_0 = -0''.302 \sin S + 0''.065 \cos S + 0''.041 \sin J + 0''.563 \cos J.$$

It will be observed that in the perturbations of the longitude by Jupiter and Saturn we have neglected a number of small terms, the coefficients of the four largest of which are each about 0''.03. The probable error in the theory produced by this neglect is 0''.04, and it was judged best, therefore, not to encumber it with them. But, should any one wish to include their effect, it can readily be calculated. Then, we have

Provisional longitude of Neptune, referred to the mean equinox

= Precession, + Longitude in pure elliptic orbit, from elements page 39

$$+\delta l + (P_{s.1} + 2\,\delta k) \sin l + (P_{c.1} - 2\,\delta h) \cos l + P_{s.2} \sin 2l + P_{c.2} \cos 2l + \delta v_0$$

+ Reduction to ecliptic.

Common logarithm of the radius vector

= Log. radius vector in elliptic orbit

$$-.0005920 + \delta a + (R_{s.1} - M\delta h) \sin l + (R_{c.1} - M\delta k) \cos l + \delta r_0.$$

Latitude =

Latitude in elliptic orbit (the longitude being increased by the perturbations),

$$+ (B_{s.1} + \delta q) \sin v + (B_{c.1} - \delta p) \cos v + \delta\beta_0.$$

l is the mean longitude of Neptune, and v its true longitude in orbit, referred to the mean equinox of 1850.0.

§ 20. These formulæ give the following heliocentric positions of Neptune:

Heliocentric co-ordinates of Neptune, referred to the mean equinox of date, for each 180th day, Greenwich mean noon.

Date.		Longitude.	Latitude.	log *r*.
		° ′ ″	° ′ ″	
1795,	May 9,	215 5 20.12	+1 47 59.80	1.4817427
1846,	Jan. 21,	325 28 41.54	—0 28 26.90	1.4774075
	July 20,	326 33 58.15	0 30 23.48	3215
1847,	Jan. 16,	327 39 13.82	0 32 19.44	2356
	July 15,	328 44 28.74	0 34 14.63	1510
1848,	Jan. 11,	329 49 43.21	0 36 9.15	1.4770685
	July 9,	330 54 57.50	—0 38 2.92	1.4769892
1849,	Jan. 5,	332 0 11.98	0 39 55.91	9135
	July 4,	333 5 27.14	0 41 48.06	8420
	Dec. 31,	334 10 43.37	0 43 39.37	7742
1850,	June 29,	335 16 1.07	0 45 29.78	7104
	Dec. 26,	336 21 20.50	—0 47 19.25	6503
1851,	June 24,	337 26 42.10	0 49 7.76	5935
	Dec. 21,	338 32 6.10	0 50 55.27	5396
1852,	June 18,	339 37 32.77	0 52 41.71	4880
	Dec. 15,	340 43 2.18	0 54 27.04	4383
1853,	June 13,	341 48 34.62	—0 56 11.23	3895
	Dec. 10,	342 54 10.02	0 57 54.26	3405
1854,	June 8,	343 59 48.28	0 59 36.04	2907
	Dec. 5,	345 5 29.19	1 1 16.58	2395
1855,	June 3,	346 11 12.34	1 2 55.79	1856
	Nov. 30,	347 16 57.47	—1 4 33.67	1281
1856,	May 28,	348 22 43.95	1 6 10.17	0671
	Nov. 24,	349 28 31.22	1 7 45.23	1.4760028
1857,	May 23,	350 34 18.63	1 9 18.83	1.4759357
	Nov. 19,	351 40 5.78	1 10 50.93	8652
1858,	May 18,	352 45 52.17	—1 12 21.48	7918
	Nov. 14,	353 51 37.46	1 13 50.50	7185
1859,	May 13,	354 57 21.65	1 15 17.89	6454
	Nov. 9,	356 3 4.60	1 16 43.65	5739
1860,	May 7,	357 8 46.65	1 18 7.79	5049
	Nov. 3,	358 14 27.84	—1 19 30.24	4394
1861,	May 2,	359 20 8.66	1 20 50.95	3779
	Oct. 29,	0 25 49.45	1 22 9.96	3210
1862,	April 27,	1 31 30.52	1 23 27.19	2691
	Oct. 24,	2 37 12.30	1 24 42.64	2220
1863,	April 22,	3 42 55.16	—1 25 56.24	1797
	Oct. 19,	4 48 39.57	1 27 7.97	1423
1864,	April 16,	5 54 25.74	1 28 17.84	1093
	Oct. 13,	7 0 14.10	1 29 25.77	0801
1865,	April 11,	8 6 4.93	—1 30 31.79	1.4750547

From these heliocentric positions are concluded the following *apparent* geocentric positions, corrected for aberration, for the dates of the normal places to be given in the next chapter.

Date.	Geocentric Longitude.	Geocentric Latitude.	Date.	Geocentric Longitude.	Geocentric Latitude.
	° ′ ″	° ′ ″		° ″	° ′ ″
1795, May 9,	214 37 19.1	+1 50 34.4	1856, Aug. 8,	349 54 3.3	−1 8 44.8
1846, Oct. 14,	325 31 34.9	−0 31 56.0	Sept. 13,	348 58 37.4	1 9 27.2
Nov. 14,	325 23 23.2	0 31 44.0	Oct. 26,	347 56 48.8	1 9 8.2
1847, July 26,	329 41 22.0	−0 35 25.9	Nov. 17,	347 40 53.1	1 8 35.3
Aug. 17,	329 7 18.3	0 35 47.7	1857, Aug. 13,	352 5 16.5	−1 12 6.2
Oct. 8,	327 52 10.4	0 35 58.8	Sept. 21,	351 4 2.0	1 12 46.1
Nov. 18,	327 36 56.3	0 35 38.6	Oct. 24,	350 16 9.6	1 12 28.3
1848, July 25,	331 58 5.0	−0 39 22.8	Dec. 8,	349 54 1.4	1 11 11.8
Aug. 29,	331 3 15.8	0 39 55.2	1858, Aug. 18,	354 23 52.6	−1 15 21.0
Oct. 6,	330 8 55.3	0 39 57.8	Sept. 23,	353 19 17.2	1 15 56.6
Nov. 17,	329 49 22.4	0 39 32.8	Oct. 28,	352 28 49.2	1 15 34.8
1849, Sept. 1,	333 15 38.6	−0 43 52.7	Dec. 12,	352 8 46.7	1 14 11.8
Oct. 15,	332 15 32.4	0 43 49.2	1859, Aug. 21,	356 30 2.9	−1 18 25.8
Nov. 25,	332 4 16.7	0 43 17.1	Sept. 23,	355 37 44.6	1 19 0.0
1850, Aug. 28,	335 39 38.5	−0 47 42.7	Nov. 8,	354 34 29.4	1 18 23.0
Oct. 15,	334 31 9.5	0 47 41.1	Dec. 14,	354 22 52.5	1 17 9.3
Nov. 20,	334 15 23.8	0 47 9.5	1860, Aug. 20,	358 47 48.1	−1 21 17.7
1851, Sept. 2,	337 48 58.1	−0 51 33.7	Sept. 21,	357 54 28.1	1 21 56.4
Oct. 14,	336 48 10.9	0 51 30.0	Oct. 31,	356 58 22.4	1 21 32.2
Nov. 20,	336 28 31.0	0 50 54.1	Dec. 13,	356 36 24.7	1 20 4.6
1852, Aug. 7,	340 46 11.0	−0 54 51.6	1861, Aug. 22,	1 2 42.4	−1 24 6.0
Sept. 5,	340 0 10.3	0 55 19.5	Sept. 18,	0 21 7.6	1 24 42.2
Oct. 12,	339 5 43.0	0 55 14.8	Oct. 30,	359 16 38.4	1 24 24.6
Nov. 28,	338 43 23.4	0 54 23.3	Dec. 7,	358 50 3.3	1 23 7.9
1853, Sept. 1,	342 24 47.7	−0 58 55.9	1862, Aug. 24,	3 17 34.7	−1 26 46.7
Oct. 15,	341 19 0.2	0 58 52.4	Sept. 23,	2 31 7.7	1 27 25.7
Nov. 24,	340 56 3.0	0 58 4.7	Nov. 6,	1 25 26.1	1 26 57.3
1854, Aug. 30,	344 46 17.8	−1 2 27.5	Dec. 15,	1 4 10.0	1 25 30.0
Sept. 24,	344 5 33.2	1 2 37.0	1863, Aug. 28,	5 29 46.2	−1 29 23.7
Oct. 27,	343 23 17.3	1 2 15.3	Sept. 27,	4 42 49.6	1 30 0.3
Dec. 5,	343 12 2.8	1 1 19.1	Nov. 17,	3 30 58.0	1 29 12.5
1855, Aug. 10,	347 34 54.2	−1 5 28.0	Dec. 12,	3 18 18.2	1 28 12.4
Sept. 8,	346 49 57.0	1 6 1.8	1864, Aug. 7,	8 9 20.6	−1 30 53.7
Oct. 22,	345 45 6.2	1 5 50.1	Oct. 1,	6 52 56.7	1 32 27.0
Nov. 29,	345 24 25.3	−1 4 54.8	Nov. 12,	5 51 37.8	1 31 49.1
			Dec. 17,	5 32 27.7	−1 30 23.1

The next step is to deduce positions of Neptune from observations, in order to compare them with the above theoretical positions.

CHAPTER III.

DISCUSSION OF THE OBSERVATIONS OF NEPTUNE.

§ 21. DURING the four years following the discovery of Neptune, observations of this planet, both meridian and extra-meridian, were very numerous. If the results of all these observations were free from constant errors, and, therefore, strictly comparable both with themselves and with subsequent observations, their combination would give very accurate positions of the planet. Unfortunately, however, we cannot assume that observations of different kinds, made at different observatories, are strictly comparable, nor have we, in many cases, the data for reducing them to a common standard.

Let us consider, for instance, the meridian observations. Under the title of "Meridian Observations of Neptune," we find in astronomical periodicals series of observed Right Ascensions and Declinations. But right ascensions and declinations can never be really observed with any instrument. Only times of transit, and the readings of micrometers and other instruments, are really observed. The right ascensions and declinations of the planet are concluded from the observations, by the aid of a great number of subsidiary data, some relating to the stars, others to the instrument. Respecting these data we have, in most cases, absolutely no information whatever. But a knowledge of some of them, at least, is indispensable. Even if we grant that the instrumental errors are in all cases perfectly known for every observation, we still do not know either the names or the assumed right ascensions of the stars used in determining clock errors. Hence we cannot use the results, because the right ascensions given in standard catalogues not unfrequently differ by a second of space.

The declinations of the planet are sometimes determined by comparison with standard stars, sometimes by measures of nadir distance, combined with the latitude of the observatory. The Paris observations are reduced by the former method; those of most other observatories, by the latter. Using the latter method, it would naturally be supposed that the declinations from the observations of all observatories of which the latitudes are well determined ought to agree. But such is far from being the case. Compare, for instance, the declinations of fundamental stars concluded from observations with the great transit circle at Greenwich with those in the Tabulæ Reductionum of Wolfers, and we shall find that for stars more than 45° from the pole, the Greenwich positions are systematically nearly a second south of Wolfers', an amount greater than the probable error of a single isolated observation. We cannot impeach either authority. Wolfers' positions depend on such authorities as Pond, Struve, Argelander, Henderson, Airy, and Bessel. The conscientious care bestowed on the reduction of the Greenwich observations would seem to render their results unimpeachable. Besides, from a comparison of Winnecke's observations of his "Mars Stars" in

1862 with those of Greenwich, it would seem that the meridian circle of Pulkowa gives declinations an entire second farther south than those of the great transit circle; so that had the Pulkowa instrument been employed on fundamental stars, their declinations would have been 2″ less than Wolfers'. On the other hand, the Cambridge (Eng.) mural circle places the fundamental stars even farther north than Wolfers, and the Washington mural nearly as far north.

It is foreign to our present purpose to speculate upon the causes of these discrepancies; we are concerned only with their existence and amount. Their existence renders it absolutely necessary to correct the declinations as well as the right ascensions in order to reduce them to a common standard; and no observations have been used unless data for these corrections could be obtained.

This rule necessitates the entire rejection of nearly all the vast mass of observations on which Walker's theory was founded. In the case of the micrometric comparisons, no sufficient data seem to exist for determining the positions of the comparison stars; the results are, therefore, heterogeneous in their character. However valuable they might have been when made, it will not be admissible to combine them with the fifteen years of meridian observations made since. Micrometric observations were almost given up after 1850, and the planet was left to be followed by the meridian instruments of the larger observatories. The superior accuracy of this class of observations may be inferred from the fact that the comparatively small error in Walker's radius vector is made evident by them even during the period of construction of Walker's theory.

A similar remark applies to the meridian observations. Four years of observations made at a great number of observatories may be indiscriminately combined on the supposition that the systematic as well as the accidental errors will destroy each other, particularly if each series extends through the entire period. But, as few or none of these series made at observatories able to publish any thing but their results are continued later than 1849, it will not do to assume that the mean of their systematic errors, as fixed by the standard we have assumed, would vanish.

The observations which fulfil the conditions we have indicated are made at observatories, as follows:

Ancient observations.

Paris, by Lalande, May 8 and 10 1795.

Modern observations.

Greenwich,	1846 to 1864.
Cambridge,	1846 to 1857.
Paris,	1856 to 1861.
Washington,	1846 to 1850.
Washington,	1861 to 1864.
Hamburg,	1846 to 1849.
Albany,	1861 to 1864.

§ 22. *Reduction of Lalande's two observations of Neptune, May* 8–10, 1795.

The first of these observations is found in the Comptes Rendus, tome 24, p. 667. The second is in the Histoire Céleste, p. 158, and is the eighth star of the first

column. They were made with the large mural quadrant of the observatory attached to the Military School. The Histoire Céleste does not seem to contain any definite information as to the observer or observers by whom the observations were made.

The stars of comparison which I shall select for the determination of the errors of the instrument and clock are the following:

May 8.	May 10.
β Virginis,	α Virginis,
δ Corvi,	i Virginis,
q Virginis,	λ Virginis,
ψ Virginis,	2 Libræ,
α Virginis,	μ Libræ,
h Virginis,	ξ' Libræ.
κ Virginis,	
λ Virginis,	
2 Libræ,	
ε Libræ.	

These lists, I believe, include all of Bradley's stars observed by Lalande on the dates in question within the zone of the planet, for which reliable modern positions can readily be obtained. Their positions for the year 1795 were obtained as follows. The positions given by Bessel in the Fundamenta Astronomiæ were reduced by the precessions there given to the mean equinox and equator of 1795.0. The modern positions were obtained from the Greenwich Twelve Year Catalogue, the Greenwich observations, or Rumker's Catalogue, and were also reduced to 1795.0 with Bessel's precessions. The difference of the results, being supposed due to proper motion, was divided proportionally to the time, and the concluded true position for 1795 obtained. As Lalande's observations are subject to errors of several seconds, any farther refinement in investigating the positions of the stars would be a waste of labor. In the following table is exhibited the position of the star at the two epochs, referred to the mean equinox and equator of 1795.0, with the modern authorities, and the concluded mean positions for 1795.0:

Star.	R. A., 1755.	Seconds of R. A., modern epoch.	Year of modern epoch.	Modern authority.	Dec. 1755.	Seconds of modern Dec.	For 1795.0. Concluded. R. A.	For 1795.0. Concluded. Dec.
	h. m. s.	s.			° ′ ″	″	h. m. s.	′ ″
δ Corvi,	12 19 16.36	15.87	1850	12 Y. C.	— 15 22 13.2	28.7	12 19 16.15	— 15 22 19.7
q Virginis,	12 23 12.94	12.50	1840	12 Y. C.	— 8 19 9.8	9.7	12 23 12.73	— 8 19 9.8
ψ Virginis,	12 43 42.28	42.40	1840	12 Y. C.	— 8 25 17.5	21.9	12 43 42.34	— 8 25 19.5
i Virginis,	13 15 54.88	54.32	1859	Gr. Obs. 1859	— 11 38 6.2	10.0	13 15 54.66	— 11 38 7.7
h Virginis,	13 22 11.38	11.30	1845	12 Y. C.	— 9 6 11.5	15.1	13 22 11.34	— 9 6 13.1
κ Virginis,	14 1 58.50	58.85	1840	12 Y. C.	— 9 18 39.3	37.3	14 1 58.67	— 9 18 38.4
λ Virginis,	14 8 2.29	2.41	1840	12 Y. C.	— 12 25 11.2	9.7	14 8 2.35	— 12 25 10.5
2 Libræ,	14 12 25.07	25.17	1842	Rumker.	— 10 46 5.9	11.7	14 12 25.12	— 10 46 8.6
μ Libræ,	14 38 6.33	6.19	1845	12 Y. C.	— 13 17 6.0	8.4	14 38 6.27	— 13 17 7.1
ξ' Libræ,	14 43 16.60	16.11	1842	Rumker.	— 11 3 6.2	6.3	14 43 16.37	— 11 3 6.2
ε Libræ,	15 13 . 6.60	6.14	1842	Rumker.	— 9 34 22.7	57.6	15 13 6.39	— 9 34 29.5

The above places were reduced to the dates of observation with the constants of the Tabulæ Regiomontanæ.

The apparent positions of β Virginis and α Virginis are derived from the same work, correcting the Declination of the latter by $+ 0''.60$. The former is not used for index error, owing to its distance from the zone of Neptune.

Intervals of wires.

On attempting to test the wire intervals of Lalande, H. C., p. 576, the interval of the third wire was found to exhibit well-marked systematic discrepancies. The observations of May 10 concur very well in indicating a diminution of $0^s.10$; and this correction has been applied to Lalande's intervals. The interval for wire 1 has not been changed.

Deviation of instrument.

The next quantity required is the deviation of the instrument from the circle of Right ascension of the planet. On using Lalande's value of this correction, stars of different altitudes, even in the zone of observation, gave inadmissible discrepancies. It is found necessary to reduce the value to less than half. This will be readily seen from the table below.

Clock error, &c.

The following tables give, for each star and each date—

The number of wires observed, $\frac{1}{2}$ meaning a doubtful observation.

The concluded time of transit over the middle wire.

Lalande's correction to this time for deviation of the middle wire, this deviation being supposed to vanish at the circle reading for Neptune, viz.: 60° 7′.

The correction for deviation actually applied, derived from the comparison of clock corrections given by β Virginis and δ Corvi.

Seconds of apparent R. A. of star.

The clock correction, using Lalande's deviation.

The clock correction, using the concluded deviation.

The weight assigned to the result for clock correction, depending on the number of wires, and the proximity of the star to the planet.

For the second observation the deviation is of less importance than for the first, the planet being near the middle of the zone, and the mean of the corrections, therefore, very small.

Name of star.	N.	T.	D′.	D.	R. A.	C′.	C.	W.
1795, May 8.								
		h. m. s.	s.	s.	s.	s.	s.	
β Virginis,	1½	11 39 42.67	—3.80	—1.90	1.86	22.99	21.09	0
δ Corvi,	2	12 18 55.85	+0.81	+0.40	17.31	20.65	21.06	1
q Virginis,	2	12 22 52.10	—0.60	—0.30	13.84	22.34	22.04	2
ψ Virginis,	1	12 43 22.00	—0.58	—0.29	43.53	22.11	21.82	1
α Virginis,	3	13 14 4.23	—0.24	—0.12	26.09	22.10	21.98	4
h Virginis,	2	13 21 50.55	—0.44	—0.22	12.67	22.56	22.34	3
κ Virginis,	3	14 1 38.57	—0.40	—0.20	0.09	21.92	21.72	6
λ Virginis,	3	14 7 41.80	+0.20	+0.10	3.81	21.81	21.91	6
2 Libræ,	2	14 12 4.45	—0.11	—0.05	26.53	22.19	22.13	5
ε Libræ,	1½	15 12 46.07	—0.34	—0.17	7.89	22.16	21.99	2
May 10.								
α Virginis,	2	13 14 3.55	—0.24	—0.12	26.09	22.78	22.66	1
i Virginis,	2	13 15 32.90	+0.06	+0.03	55.99	23.03	23.06	1
λ Virginis,	2	14 7 41.20	+0.22	+0.11	3.82	22.40	22.51	2
2 Libræ,	2	14 12 3.40	—0.11	—0.06	26.53	23.24	23.19	2
μ Libræ,	1	14 37 45.10	+0.39	+0.20	7.79	22.30	22.49	1
ξ′ Libræ,	2	14 42 54.95	—0.06	—0.03	17.88	22.99	22.96	1

We have then

	May 8.	May 10.
Clock time of transit of planet,	14 11 36 .50	14 11 23 .50
Correction for clock and instrument,	+21 .94	+22 .82
Concluded apparent Right Ascension,	14 11 58 .44	14 11 46 .32
or,	212° 59′ 36″.6	212° 56′ 34″.8

Declinations.

We use Bessel's refractions. For the height of the Barometer, and the temperature of the air, we have:

			in.	°	
May 8, ...	Bar. = 28 pou. 6 l.	= 30.37 Eng.;	$T = 13$	Reau. = 61.2 Fah.
May 10. Beginning of observations, .	Bar. = 28 pou. 3.1 l.	= 30.12 Eng.;	$T = 13.7$	Reau. = 62.8 Fah.
End " .	Bar. = 28 pou. 1.5 l.	= 30.07 Eng.;	$T = 13$	Reau. = 61.2 Fah.

The equatorial points on the circle are concluded as follows:

May 8.					May 10.				
Name of star.	Observed Z. Dist.	Refraction.	Declination.	Equatorial point.	Name of star.	Observed Z. Dist.	Refraction.	Declination.	Equatorial point.
	° ′ ″	′ ″		48° 49′ ″		° ″	′ ″	° ′ ″	48° 49′ ″
δ Corvi,	64 9 52	1 57.3	—15 22 29.0	20.3	α Virginis,	58 53 2	1 34.2	—10 5 17.3	18.9
q Virginis,	57 7 11	1 28.0	—8 19 17.5	21.5	i Virginis,	60 25 54	1 40.2	—11 38 14.2	20.0
ψ Virginis,	57 13 17	1 28.4	—8 25 26.6	18.8	λ Virginis,	61 12 50	1 43.5	—12 25 15.4	18.1
α Virginis,	58 53 0	1 34.3	—10 5 17.3	17.0	2 Libræ,	59 33 59	1 36.8	—10 46 13.3	22.5
h Virginis,	57 54 5	1 30.6	—9 6 19.0	16.6	μ Libræ,	62 4 43	1 47.3	—13 17 10.7	19.6
κ Virginis,	58 6 37	1 31.4	—9 18 43.0	25.4	ξ′ Libræ,	59 50 50	1 37.9	—11 3 9.5	18.4
λ Virginis,	61 12 43	1 43.5	—12 25 15.2	11.3					
2 Libræ,	59 33 57	1 36.9	—10 46 13.1	20.8					
ε Libræ,	58 22 13	1 32.4	—9 34 31.7	13.7					

Taking the means of the separate results for equatorial point, we have, for the apparent declinations of Neptune—

	May 8.	May 10.
	° ′ ″	° ′ ″
Observed circle reading,	60 8 17	60 7 19
Refraction,	1 39.0	1 39.0
Corrected circle reading,	60 9 56.0	60 8 58.0
Equatorial point,	48 49 18.4	48 49 19.6
Apparent declination,	—11 20 37.6	—11 19 38.4

§ 23. Probable errors of these positions.

So far as we can judge from the discordance of the clock errors, and equatorial points derived from the several stars, the probable error of a single observation over a single wire in right ascension would appear to be about $0^s.27$, and the probable error of a single observed zenith distance about $2''.2$. The agreement of the difference of the two observations with the computed motion of the planet shows that neither observation is affected with any abnormal error. We conclude, therefore, that the probable error of the normal place derived from the two observations is about $2''.8$ in R. A. and $1''.5$ in declination.

Notwithstanding the magnitude of these probable errors, the observations will be very valuable during the remainder of the present century, owing to the weight with which they enter into the expressions of the elements. But in the twentieth century the observations made after 1846 will enable astronomers to compute the position of the planet in 1795 with a much higher degree of accuracy than Lalande could observe it.

A similar remark applies to Lamont's accidental zone observations in 1845. Valuable during the first two or three years, they afterward ceased to be so, because the theory soon became more accurate than the observation for an epoch so near the time of optical discovery. Had they been made in 1820, they would still have been valuable.

Reduction of the modern observations.

§ 24. The modern observations will be treated in the following manner. The observations of each year will be divided into four groups, according to the time of culmination of the planet. The first group will include all observations made after

	h. m. h. m.
	13 30 m. t.
Second, between	10 30 and 13 30.
Third, "	7 30 and 10 30.
Fourth, all made before	7 30.

The mean correction derived from each group will at first be regarded as the true correction applicable to the mean of the times of observation. This involves the supposition that the error of the ephemeris is changing uniformly during each series of observations. If we could compare with an ephemeris of the heliocentric

place of the planet, this hypothesis would be sufficiently near the truth for an entire year or more. But the error of geocentric place would be subject to an annual period though the errors of the heliocentric place should be invariable. Let us estimate the error of the hypothesis in question. Put

r = radius vector of Neptune.
D = difference of longitude of Sun and Neptune.
δv, δr, errors of heliocentric longitude and radius vector.

Then the errors of geocentric longitude will be, approximately,

$$\delta v\left(1+\frac{\cos D}{r}\right)+\frac{\delta r}{r^2}\sin D.$$

Of this expression the part

$$\frac{\delta v}{r}\cos D+\frac{\delta r}{r^2}\sin D$$

will not be regularly progressive, but will change with the sine and cosine of D, the period of which is about 368 days.

The integral of this expression gives for the mean value of the error, while D is increasing from D_0 to D_1,

$$\frac{\delta v}{r}\cdot\frac{\sin D_1-\sin D_0}{D_1-D_0}-\frac{\delta r}{r^2}\cdot\frac{\cos D_1-\cos D_0}{D_1-D_0}.$$

By putting

$$D=\frac{D_1-D_0}{2}, \qquad \delta=D_1-D=D-D_0,$$

and developing according to powers of δ, this expression becomes

$$\frac{\delta v}{r}\cos D\left(1-\frac{\delta^2}{6}\right)+\frac{\delta r}{r^2}\sin D\left(1-\frac{\delta^2}{6}\right).$$

This, plus the error of heliocentric longitude, is the mean error which will be given by a series of observations equally scattered through a period $\pm\,\delta$ on each side of the mean epoch D. But what we really want is the error at the mean epoch itself; that is,

$$\delta v+\frac{\delta v}{r}\cos D+\frac{\delta r}{r^2}\sin D;$$

so that we must correct the mean error actually found by the quantity

$$\frac{\delta^2}{6}\left(\frac{\delta v}{r}\cos D+\frac{\delta r}{r^2}\sin D\right),$$

or, since δ is generally about 1^h_{4}, and r about 30,

$$.027\left(\frac{\delta v}{30}\cos D+\frac{\delta r}{900}\sin D\right)$$

The maximum value of δv being less than 30″, the first term will be entirely neglected. The value of δr sometimes amounts to .018, so that the correction arising from the second term may sometimes amount to 0″.11. We shall, therefore, take account of it in a few cases.

The ephemeris which will be compared with observation in order to deduce normal places of the planet will be the same with which the Greenwich observations are compared, namely, Walker's ephemeris until the year 1854, and Kowalski's ephemeris in subsequent years. It will be remembered, however, that these ephemerides are used only for the purpose of obtaining normal places, and in order to save the trouble of comparing every individual observation with the provisional theory.

§ 25. *Mean corrections of the Ephemeris of Neptune given by observations at the different observatories, without correction for systematic differences.*

GREENWICH.

Date.	R. A.	Dec.	No.
	s	″	
1846, Oct. 14,	−0.050	+0.48	12
Nov. 16,	−.070	+0.55	7
1847, July 26,	−.150	+2.05	4
Aug. 20,	−.097	+2.23	10
Oct. 3,	−.145	+1.43	10
Nov. 24,	−.056	+1.76	8
1848, July 28,	−.062	+1.16	4
Aug. 31,	+.002	−0.20	8
Oct. 7,	−.104	+0.01	14
Nov. 16,	+.022	+0.11	4
1849, Sept. 3,	−.027	+0.65	6
Oct. 17,	+.080	+1.70	8
Nov. 28,	−.060	+1.96	5
1850, Aug. 27,	−.079	−1.00	13
Oct. 16,	+.040	+0.20	13
Nov. 24,	+.020	−0.52	16
1851, Sept. 1,	−.162	−1.07	16
Oct. 12,	+.060	−0.97	4
Nov. 9,	−.040	−1.94	5
1852, Aug. 7,	−.260	−2.34	5
Sept. 11,	−.160	−2.44	10
Oct. 12,	−.140	−3.36	10
Nov. 22,	−.080	−2.27	5
1853, Sept. 1,	−.256	−2.59	14
Oct. 11,	−.177	−2.93	16
Nov. 19,	−.160	−2.71	3
1854, Aug. 30,	−.420	−3.60	13
Sept. 24,	−.370	−3.94	11
Oct. 27,	−.310	−3.68	7
Dec. 5,	−.300	−4.36	4
1855, Aug. 10,	−.189	−0.84	7
Sept. 8,	−.046	−0.06	16
Oct. 22,	+.183	+0.80	6
Nov. 29,	+.177	+1.51	6
1856, Aug. 8,	−.220	−1.06	10
Sept. 13,	−.080	−1.06	7
Oct. 26,	+.076	+1.41	9
Nov. 17,	+.123	+1.67	6
1857, Aug. 14,	−.356	−2.43	5
Sept. 22,	−.130	−0.50	12
Oct. 24,	+.030	+0.29	5
Dec. 5,	+.130	+0.16	10
1858, Aug. 18,	−.391	−1.74	14
Sept. 24,	−.260	−1.81	13
Oct. 25,	−.206	−1.00	16
Dec. 10,	−.058	−0.76	11
1859, Aug. 19,	−.500	−3.27	9
Sept. 28,	−.446	−3.11	17
Nov. 3,	−.315	−2.56	15
Dec. 16,	−.328	−1.46	10

CAMBRIDGE.

Date.	R. A.	No.	Dec.	No.
	s		″	
1846, Oct. 13,	−0.014	10	+1.51	8
Nov. 7,	.000	14	+1.43	15
1847, July 27,	−.062	5	+2.30	4
Aug. 22,	−.090	18	+2.25	17
Oct. 8,	−.100	14	+2.18	13
Nov. 20,	−.022	13	+0.66	14
1848, July 22,	−.056	8	+0.88	8
Aug. 27,	−.048	19	+1.85	19
Oct. 9,	−.018	16	+0.75	17
Nov. 19,	+.009	10	+1.05	11
1849, Aug. 21,	−.087	15	+0.72	18
Oct. 15,	−.038	16	−0.17	16
Nov. 22,	+.090	11	+0.21	2
1850, Aug. 29,	−.089	10	+0.48	11
Oct. 16,	+.011	14	−0.13	15
Nov. 23,	+.039	12	+0.14	13
1851, Sept. 4,	−.054	18	−1.62	19
Oct. 17,	+.028	11	−1.61	10
Nov. 28,	+.014	9	−1.91	10
1852, Aug. 29,	−.037	15	−1.53	15
Oct. 11,	−.048	10	−2.52	11
Dec. 4,	+.038	7	−2.99	7
1853, Sept. 2,	−.048	4	−2.48	5
Oct. 24,	−.134	13	−3.04	13
Nov. 27,	+.031	11	−2.53	12
1854, Sept. 4,	−.314	11	−3.59	12
Oct. 11,	−.273	15	−4.38	3
Nov. 24,	−.165	4	−5.17	8
	s		″	
1855, Sept. 8,	−0.046	12	+0.48	9
Oct. 12,			+0.50	6
Dec. 10,	+0.206	9	+3.07	7
1856, Sept. 12,	−0.099	9	+0.05	8
Oct. 29,	+0.120	8	+1.73	7
Nov. 28,	+0.164	5	+2.50	5
1857, Sept. 14,	−0.080	9	−0.83	12
Oct. 25,	+0.104	5	−0.34	5
Dec. 11,	+0.175	8	+0.16	9

GREENWICH (Cont.).

Date.		R. A.	Dec.	No.
		s	"	
1860,	Aug. 20,	—0.760	—5.02	4
	Sept. 20,	—0.685	—3.86	13
	Oct. 31,	—0.662	—3.38	15
	Dec. 13,	—0.630	—4.73	2
1861,	Aug. 22,	—0.940	—5.41	4
	Sept. 18,	—0.861	—5.62	16
	Oct. 30,	—0.861	—5.18	12
	Dec. 7,	—0.993	—5.14	7
1862,	Aug. 24,	—1.12	—7.20	6
	Sept. 25,	—1.162	—6.76	13
	Nov. 4,	—1.139	—7.00	11
	Dec. 17,	—1.18	—7.06	4
1863,	Aug. 28,	—1.585	—9.91	2
	Sept. 23,	—1.461	—8.62	9
	Nov. 12,	—1.388	—8.33	4
	Dec. 15,	—1.375	—8.41	2
1864,	Oct. 3,	—1.680	—11.03	8
	Nov. 8,	—1.639	—10.52	7

WASHINGTON (Walker's Eph.).

Date.		R. A.	No.	Dec.	No.
		s			
1846,	Nov. 9,	+0.096	10	+2.16	7
1847,	Aug. 23,	—0.205	15	+2.27	6
	Oct. 14,	—0.023	9	+1.80	7
	Nov. 8,	+0.052	5	+1.96	3
1848,	Aug. 30,	—0.076	10	+0.80	14
	Oct. 2,	—0.121	12	+1.71	10
1849,	Sept. 11,	—0.154	5		
	Oct. 12,	—0.014	8	+1.06	6
1850,	Oct. 13,	—0.032	9	+0.94	25
	Nov. 11,	—0.068	5	+1.25	13
1861,	Oct. 29,	—1.695	6	—8.70	5
	Dec. 16,	—1.593	11	—8.64	10
1862,	Sept. 23,	—2.000	2		
	Nov. 14,	—1.860	3	—9.85	2
	Dec. 12,	—1.827	6	—10.9	1
1863,	Oct. 13,	—2.22	3	—13.82	5
	Nov. 12,	—2.19	3	—13.52	6
	Dec. 8,	—2.054	5	—12.76	7
1864,	Aug. 7,	—2.69	3	—15.5	4
	Nov. 17,	—2.52	5	—14.6	12
	Dec. 20,	—2.38	5	—14.0	2

ALBANY (Kowalski).

Date.		R. A.	No.	Dec.	No.
		s		"	
1861,	Sept. 1,	—0.778	5	—6.02	5
	Nov. 11,	—0.825	8	—5.42	8
	Dec. 14,	—0.847	9	—5.44	9

PARIS (Walker).

Date.		R. A.	No.	Dec.	No.
		s		"	
1856,	Sept. 14,	—0.669	12	—3.96	8
	Oct. 25,	—0.606	14	—3.91	15
	Dec. 21,	—0.470	2		
1857,	Sept. 19,	—0.768	10	—4.98	10
	Oct. 25,	—0.825	13	—5.95	12
	Dec. 14,	—0.729	7	—6.13	7

PARIS (Kowalski).

Date.		R. A.	No.	Dec.	No.
		s		"	
1858,	Sept. 21,	—0.291	18	—1.14	17
	Oct. 27,	—0.235	19	—0.76	19
1859,	Aug. 23,	—0.630	5	—2.64	5
	Sept. 23,	—0.474	9	—2.50	8
	Nov. 17,	—0.337	11	—1.92	14
	Dec. 7,	—0.430	2	—2.20	2
1860,	Sept. 29,	—0.608	6	—3.75	6
	Oct. 31,	—0.618	12	—3.51	10
1861,	Sept. 28,	—0.960	7	—5.56	7
	Nov. 4,	—0.948	17	—5.52	17
	Dec. 9,	—0.890	5	—5.60	4

HAMBURG.

Date.		R. A.	Dec.	No.
		s	"	
1846,	Oct. 7,	—0.098	—1.43	9
	Nov. 21,	—0.131	+0.10	16
1847,	Aug. 22,	—0.061	—1.43	17
	Oct. 10,	—0.102	+0.10	22
	Nov. 26,	—0.051	+0.32	12
1848,	July 20,	0.000	—0.20	7
	Aug. 27,	—0.118	—2.17	15
	Oct. 4,	—0.043	—2.15	12
1849,	Sept. 3,	—0.014	—0.92	9
	Oct. 10,	—0.008	—0.50	16
	Nov. 25,	+0.071	—0.71	10

ALBANY (Walker).

Date.		R. A.	No.	Dec.	No.
		s		"	
1862,	Aug. 25,	—1.927	3	—13.05	4
	Sept. 21,	—1.905	14	—13.67	17
	Oct. 31,	—1.815	9	—13.46	10
	Dec. 17,	—1.732	6	—12.73	6
1863,	Sept. 27,	—2.228	12	—14.96	1
	Nov. 6,	—2.145	10	—15.38	9
	Dec. 14,	—2.053	6	—14.80	6
1864,	Sept. 29,	—2.490	6	—17.70	6
	Nov. 9,	—2.437	9	—16.29	9
	Dec. 14,	—2.360	3	—16.27	3

§ 26. *Corrections to the observed positions in order to render them strictly comparable with each other.*

These corrections have been derived from a comparison of the positions of the ten fundamental clock stars, from γ Aquilæ to α Ceti inclusive, given by observations at the different observatories, with the adopted standard positions. The standard right ascensions are those of Dr. Gould, prepared for the United States Coast Survey. The declinations are those of Wolfers in the "Tabulæ Reductionum," diminished by 0″.50. Both are given in the following table:

	R. A. 1850.0.	Annual var. 1850.	Dec. 1850.0.	Annual var. 1850.	Cor. to Am. Eph. R. A.	Cor. to Am. Eph. Dec.
	h. m. s.	s.	° ′ ″	″		
γ Aquilæ,	19 39 7.68	+2.853	+10 15 5.02	+8.41	+2	−2
α Aquilæ,	19 43 27.82	2.928	+ 8 28 33.45	9.13	−1	+5
β Aquilæ,	19 47 56.67	2.948	+ 6 2 8.68	8.62	+3	+5
α^2 Capricorni,	20 9 43.69	3.335	−13 0 20.90	10.77	+2	+13
α Aquarii,	21 58 4.68	3.084	− 1 2 47.41	17.28	+4	+8
α Pegasi,	22 57 17.50	2.983	+14 23 57.36	19.30	0	+1
α Andromedæ,	0 0 38.57	3.085	+28 15 43.72	19.91	+2	0
γ Pegasi,	0 5 30.99	3.081	+14 20 57.67	20.04	+2	+1
α Arietis,	1 58 43.64	3.364	+22 44 1.97	17.29	+1	0
α Ceti,	2 54 26.58	+3.130	+ 3 29 52.45	+14.42	+6	+5

In reducing the Albany observations, it was found advisable to add ω Piscium to the number of standard stars for determining these corrections. Its assumed position is

R. A. 1860.0.	Declination 1860.0.
23 51 7.42	+ 6° 5′ 17″.9

The observed mean right ascensions and declinations of these stars, reduced to the beginnings of the several years, have been compared with those derived from the above table, giving the result from each star a weight proportional to the number of observations when the observations were few in number, but giving each result equal weight when they were numerous. Thus the following systematic corrections have been derived:

Greenwich.	R. A.	Dec.
	s	″
1846	+0.044	−0.04
47	+0.059	−0.19
48		+0.08
49		−1.51
50		−0.52
51	−0.020	−0.22
52		+0.21
54		+0.18
55		+0.05
56	−0.038	+0.30
58	−0.015	+0.48
60	−0.005	
61	−0.003	+0.24
62		+0.63
63		+0.27
6 Y. Cat. of 1854	−0.020	
7 Y. Cat. of 1860	+0.002	

Cambridge.	R. A.	Dec.
		″
1846		−1.21
47		−0.99
48		−0.25
49	s	−0.05
50	−0.038	−0.80
51	−0.052	−1.17
52		−0.20
53		−0.75
54	−0.044	−0.73
57	−0.052	

Washington.	R. A.	Dec.
	s	″
1846	+0.034	−0.34
47	+0.057	−0.41
48	+0.036	−0.57
49 } 50 }	+0.039	−1.04
61	+0.058	−0.89
63		−0.69
64		−1.37

Paris.	R. A.	Dec.
	s	″
1856	+0.020	−0.57
58	+0.021	−0.23
60	+0.023	−0.53

Albany.	R. A.	Dec.
	s	″
1861	+0.006	+0.17
62	000	0.00
63	000	+0.78
64	000	+0.97

REMARKS ON THE PRECEDING CORRECTIONS.

Greenwich.

The corrections actually applied to the right ascensions from 1848 to 1853 have been derived by comparing the corrections on p. IV. of the introduction to the Greenwich six-year catalogue for 1854 with the corrections given by that catalogue, namely, — $0^s.020$. From 1857 to 1864 the corrections have been derived in the same way from the seven-year catalogue for 1860. The entire list of corrections is as follows:

1846,	+ $0^s.044$
47,	+ 0 .059
48,	+ 0 .052
49–55,	— 0 .010
56,	— 0 .025
57–61,	— 0 .008
62–64,	+ 0 .002

The corrections to the declination have been concluded from year to year from the table.

Cambridge.

One consistent set of adopted right ascensions having been used in the reductions of the Cambridge observations, the constant correction

$$— 0^s.046$$

has been applied to the right ascensions throughout. The declinations have been corrected as follows:

1846–47,	— $1''.12$
1848–57,	— 0 .58

Washington.

The corrections to the Washington right ascensions from 1846 to 1850 have been derived from a general comparison of twenty-five fundamental stars near the equator with the results of the Greenwich observations. The mean + $0^s.042$ has been adopted as the constant correction for those years. After 1861, no correction is needed, Dr. Gould's Right ascensions having been adopted in the reductions.

The corrections to the declinations for 1861 have been derived from those for 1862. The latter were diminished by $0''.20$ for error of nadir point, while no such correction was applied to the former.

Hamburg.

Having applied to Charles Rumker, Esq., M.A., of the observatory at Hamburg, for information respecting the data used in the reduction of the Hamburg observations of Neptune, I was informed that both right ascensions and declinations

depended on the positions of the Nautical Almanac stars. For the years 1846–47, the Nautical Almanac right ascensions require the constant correction — $0^s.003$, and in 1848–49 the correction + $0^s.049$, to reduce them to those adopted.

The declinations do not seem so easily reducible to our adopted standard. They are, therefore, not included.

All the Washington, and some of the Paris and Albany, observations having been compared with Walker's Ephemeris in years subsequent to 1855, the following corrections have been applied for differences of Ephemerides:

To Paris Corrections.

Date.	R. A.	Dec.
	s	
1856, Sept. 14,	+ 0.54	+ 4.68
Oct. 25,	+ 0.65	+ 5.75
1857, Sept. 19,	+ 0.676	+ 5.02
Oct. 25,	+ 0.80	+ 5.82
Dec. 14,	+ 0.80	+ 5.92

To Washington and Albany Corrections.

Date.	R. A.	Dec.
	s	
1861, Oct. 29,	+ 0.76	+ 5.5
Dec. 16,	+ 0.70	+ 4.9
1862, Aug. 25,	+ 0.90	+ 6.3
Sept. 21,	+ 0.85	+ 6.2
23,	+ 0.85	+ 6.2
Oct. 31,	+ 0.76	+ 5.4
Nov. 14,	+ 0.75	+ 5.0
Dec. 12,	+ 0.70	+ 5.1
17,	+ 0.70	+ 5.1
1863, Sept. 27,	+ 0.845	+ 5.9
Oct. 13,	+ 0.81	+ 5.6
Nov. 6,	+ 0.78	+ 5.3
12,	+ 0.77	+ 5.5
Dec. 8,	+ 0.73	+ 5.2
14,	+ 0.73	+ 5.2
1864, Aug. 7,	+ 0.91	+ 6.2
Sept. 29,	+ 0.87	+ 6.0
Nov. 9,	+ 0.90	+ 5.9
17,	+ 0.88	+ 5.5
Dec. 14,	+ 0.82	+ 5.8
20,	+ 0.82	+ 5.5

§ 27. The concluded corrections of the ephemeris for normal dates generally near the mean of the means have been concluded by applying to the corrections of pp. 51, 52 the following corrections:

1. Correction for systematic error given by fundamental stars.

2. Reduction, when the change of error was rapid, from the dates of the means to the dates of the normals.

3. 0.027 $\frac{\delta r}{900}$ sin D for second differences of error, when $\delta r > .01$.

4. Correction just given for difference of ephemerides.

The results are given in the following table. The small figures show the relative weights assigned to the separate results, which are, to a certain extent, a matter of judgment, but which are assigned without any reference to the magnitude of the correction itself.

Corrections to the Tabular Right Ascensions given by the different Observatories, with the Concluded Corrections and Concluded Normal Right Ascensions.

(The units are hundredths of seconds of time.)

	Gr.	Cam.	Par.	Wash.	Ham.	Concluded.	Tab. R. A.	R. A. from Observation.
							s.	h. m. s.
1846, Oct. 14,	-1_5	-6_2			-10_2	-4	55.02	21 51 54.98
Nov. 14,	-3_4	-5_3		$+14_5$	-13_2	$+1$	22.99	21 51 23.00
1847, July 26,	-9_3	-11_1				-9	1.94	22 8 1.85
Aug. 17,	-4_5	-14_3		-16_6	-6_2	-11	51.90	22 5 51.79
Oct. 8,	-9_5	-14_2		$+2_5$	-10_3	-6	3.42	22 1 3.36
Nov. 18,	0_4	-7_2		$+9_3$	-5_2	0	4.28	22 0 4.28
1848, July 25,	-1_3	-10_2			$+5_1$	-3	49.85	22 16 49.82
Aug. 29,	$+5_5$	-9_3		-3_6	-7_2	-2	21.55	22 13 21.53
Oct. 6,	-5_6	-6_2		-8_5	$+1_2$	-5	53.89	22 9 53.84
Nov. 17,	$+7_3$	-4_2				$+3$	38.40	22 8 38.43
1849, Sept. 1,	-4_3	-13_2		-12_3	$+3_2$	-7	51.43	22 21 51.36
Oct. 15,	$+7_4$	-8_2		$+3_6$	$+4_2$	$+2$	2.74	22 18 2.76
Nov. 25,	-7_3	$+4_2$		-2_1	$+12_2$	$+1$	19.06	22 17 19.07
1850, Aug. 28,	-9_5	-14_2				-10	62.59	22 31 2.49
Oct. 15,	$+3_5$	-4_2		$+1_6$		$+1$	43.45	22 26 43.46
Nov. 20,	$+1_6$	-1_2		-2_3		$+0$	42.94	22 25 42.94
1851, Sept. 2,	-17_6	-10_3				-15	15.94	22 39 15.79
Oct. 14,	$+5_3$	-2_2				$+2$	26.94	22 35 26.96
Nov. 20,	-5_3	-3_2				-4	11.92	22 34 11.88
1852, Aug. 7,	-27_3					-27	25.60	22 50 25.33
Sept. 5,	-17_3	-8_2				-15	34.10	22 47 33.95
Oct. 12,	-15_5	-9_2				-13	9.83	22 44 9.70
Nov. 28,	-9_3	-1_1				-7	44.70	22 42 44.63
1853, Sept. 1,	-26_6	-9_1				-24	40.14	22 56 39.90
Oct. 15,	-18_6	-18_4				-18	34.47	22 52 34.29
Nov. 24,	-17_2	-2_3				-8	7.41	22 51 7.33
1854, Aug. 30,	-43_5	-37_3				-41	32.47	23 5 32.06
Sept. 24,	-38_3					-38	1.25	23 3 0.87
Oct. 27,	-32_4	-33_6				-33	23.38	23 0 23.05
Dec. 5,	-31_3	-21_2				-27	40.04	22 59 39.77
1855, Aug. 10,	-20_{14}					-20	2.20	23 16 2.00
Sept. 8,	-6_{24}	-10_6				-7	16.50	23 13 16.43
Oct. 22,	$+17_{12}$					$+17$	15.60	23 9 15.77
Nov. 29,	$+17_{12}$	$+15_5$				$+16$	57.33	23 7 57.49

Corrections to the Tabular Right Ascensions given by the different Observatories, with the Concluded Corrections and Concluded Normal Right Ascensions (Cont.).

(The units are hundredths of seconds of time.)

	Gr.	Cam.	Par.	Wash.	Albany.	Concluded.	Tab. R.A.	R. A. from Observation.
							s.	h. m. s.
1856, Aug. 8,	-25_{20}					-25	41.68	23 24 41.43
Sept. 13,	-11_{14}	-14_{5}	-15_{2}			-12	17.95	23 21 17.83
Oct. 26,	$+5_{18}$	$+7_{5}$	$+6_{3}$			$+6$	28.76	23 17 28.82
Nov. 17,	$+11_{12}$	$+11_{3}$				$+11$	28.90	23 16 29.01
1857, Aug. 13,	-37_{12}					-37	50.90	23 32 50.53
Sept. 21,	-14_{24}	-5_{6}	-6_{2}			-12	6.08	23 29 5.96
Oct. 24,	$+2_{12}$	$+6_{3}$	0_{3}			$+2$	8.91	23 26 8.93
Dec. 8,	$+11_{21}$	$+13_{5}$	$+9_{2}$			$+11$	45.10	23 24 45.21
1858, Aug. 18,	-40_{28}					-40	58.46	23 40 58.06
Sept. 23,	-27_{26}		-27_{5}			-27	29.38	23 27 29.11
Oct. 28,	-20_{32}		-21_{4}			-20	22.93	23 34 22.73
Dec. 12,	-6_{22}					-6	6.89	23 33 6.83
1859, Aug. 21,	-51_{20}		-61_{1}			-52	14.82	23 49 14.30
Sept. 23,	-46_{30}		-45_{2}			-46	3.51	23 46 3.05
Nov. 8,	-32_{25}		-32_{3}			-32	9.94	23 42 9.62
Dec. 14,	-33_{20}		-41_{1}			-33	25.27	23 41 24.94
1860, Aug. 20,	-77_{10}					-77	45.29	23 57 44.52
Sept. 23,	-69_{25}		-59_{2}			-68	30.53	23 54 29.85
Oct. 31,	-67_{25}		-60_{3}			-66	3.89	23 51 3.23
Dec. 13,	-63_{5}					-63	40.91	23 49 40.28
1861, Aug. 22,	-95_{10}					-95	4.99	0 6 4.04
Sept. 18,	-87_{30}		-94_{2}		-77_{8}	-86	33.38	0 3 32.52
Oct. 30,	-87_{20}		-93_{5}	-88_{18}	-82_{8}	-88	36.27	23 59 35.39
Dec. 7,	-100_{15}		-87_{2}	-84_{33}	-84_{9}	-89	56.66	23 57 55.77
1862, Aug. 24,	-112_{15}				-103_{7}	-109	24.31	0 14 23.22
Sept. 23,	-116_{25}			-115_{6}	-105_{15}	-113	34.92	0 11 33.79
Nov. 6,	-114_{20}			-111_{18}	-105_{10}	-112	33.12	0 7 32.00
Dec. 15,	-118_{10}			-113_{15}	-103_{8}	-112	12.76	0 6 11.64
1863, Aug. 28,	-159_{5}					-159	34.01	0 22 32.42
Sept. 27,	-146_{20}			-141_{8}	-138_{10}	-144	42.64	0 19 41.20
Nov. 17,	-138_{10}			-141_{8}	-136_{8}	-140	17.57	0 25 16.17
Dec. 12,	-137_{5}			-132_{12}	-132_{8}	-133	29.48	0 14 28.15
1864, Aug. 7,				-178_{7}		-178	22.99	0 32 21.21
Oct. 1,	-168_{8}				-162_{8}	-167	44.63	0 27 42.96
Nov. 12,	-163_{15}			-164_{15}	-154_{9}	-162	58.31	0 23 56.69
Dec. 17,				-156_{15}	-154_{8}	-157	45.62	0 22 44.05

Correction to the Declinations, with the Concluded Declinations.

	Gr.	Cam.	Par.	Wash.	Albany.	Concluded.	Tab. Dec.	Concluded Dec. from Obs.
	″						″	° ′ ″
1846, Oct. 14,	$+0.4_4$	$+0.4_2$				$+0.4$	20.6	−13 31 20.2
Nov. 14,	$+0.5_3$	$+0.3_3$		$+1.8_3$		$+0.9$	54.7	13 33 53.8
1847, July 26,	$+1.9_2$	$+1.2_1$				$+1.7$	31.0	−12 8 29.3
Aug. 17,	$+2.0_4$	$+1.1_3$		$+1.8_3$		$+1.7$	48.0	12 20 46.3
Oct. 8,	$+1.2_4$	$+1.1_2$		$+1.4_3$		$+1.2$	6.2	12 47 5.0
Nov. 18,	$+1.6_3$	-0.5_3		$+1.5_1$		$+0.7$	2.6	12 52 1.9
1848, July 25,	$+1.2_2$	$+0.3_2$				$+0.8$	39.6	−11 23 38.8
Aug. 29,	$+0.1_3$	$+1.3_3$		$+0.2_4$		$+0.5$	44.9	11 43 44.4
Oct. 6,	$+0.1_5$	$+0.2_3$		$+1.1_4$		$+0.5$	3.1	12 3 2.6
Nov. 17,	$+0.2_2$	$+0.5_2$				$+0.4$	32.9	12 9 32.5
1849, Sept. 1,	-0.9_3	$+0.1_3$				-0.4	55.1	−10 59 55.5
Oct. 15,	$+0.2_3$	-0.8_3		$+0.0_3$		-0.2	33.0	11 21 33.2
Nov. 25,	$+1.4_2$	-0.4_3				$+0.3$	5.2	11 25 4.9
1850, Aug. 28,	-1.5_5	-0.1_2				-1.1	52.6	−10 10 53.7
Oct. 15,	-0.3_5	-0.7_3		-0.1_5		-0.3	59.4	10 35 59.7
Nov. 20,	-1.0_8	-0.4_3		$+0.2_4$		-0.5	15.3	10 41 15.8
1851, Sept. 2,	-1.3_8	-2.2_3				-1.6	27.6	− 9 26 29.2
Oct. 14,	-1.2_2	-2.2_2				-1.7	3.1	9 49 4.8
Nov. 20,	-2.2_2	-2.5_2				-2.4	45.9	9 55 48.3
1852, Aug. 7,	-2.1_3					-2.1	44.2	− 8 22 46.3
Sept. 5,	-2.2_4	-2.1_3				-2.2	37.0	8 40 39.2
Oct. 12,	-3.1_4	-3.1_2				-3.1	5.7	9 1 8.8
Nov. 28,	-2.1_3	-3.6_2				-2.7	42.2	9 8 44.9
1853, Sept. 1,	-2.4_5	-3.1_1				-2.5	53.0	− 7 48 55.5
Oct. 15,	-2.7_5	-3.6_3				-3.1	58.4	8 14 1.5
Nov. 24,	-2.5_1	-3.1_3				-2.9	58.5	8 22 1.4
1854, Aug. 30,	-3.4_4	-4.2_3				-3.7	39.1	− 6 57 42.8
Sept. 24,	-3.8_4					-3.8	32.9	7 13 36.7
Oct. 27,	-3.5_3	-5.1_1				-3.9	30.6	7 29 34.5
Dec. 5,	-4.1_2	-5.7_2				-4.9	58.1	7 33 3.0
1855, Aug. 10,	-0.8_3					-0.8	54.9	− 5 54 55.7
Sept. 8,	0.0_5	-0.1_2				0.0	59.8	6 12 59.8
Oct. 22,	$+0.9_3$	$+0.2_1$				$+0.7$	4.1	6 38 3.4
Nov. 29,	$+1.6_3$	$+2.5_2$				$+2.0$	15.2	− 6 45 13.2

CORRECTION TO THE DECLINATIONS, WITH THE CONCLUDED DECLINATIONS (Cont.).

	Gr.	Cam.	Par.	Wash.	Albany.	Concluded.	Tab. Dec.	Concluded Dec. from Obs.
	″						″	° ′ ″
1856, Aug. 8,	-0.8_4					-0.8	21.4	-5 3 22.2
Sept. 13,	-0.7_3	-0.5_2	$+0.1_2$			-0.4	48.3	5 25 48.7
Oct. 26,	$+1.7_4$	$+1.1_2$	$+1.3_5$			$+1.4$	45.4	5 49 44.0
Nov. 17,	$+2.0_3$	$+1.7_1$				$+1.9$	29.2	5 55 27.3
1857, Aug. 13,	-2.0_2					-2.0	40.8	-4 14 42.8
Sept. 21,	-0.1_6	-1.3_3	-0.4_2			-0.5	29.9	4 39 30.4
Oct. 24,	$+0.7_2$	-0.9_1	-0.5_3			-0.2	6.4	4 58 6.6
Dec. 8,	$+0.6_4$	-0.4_2	-0.6_2			0.0	39.0	5 5 39.0
1858, Aug. 18,	-1.3_5					-1.3	44.6	-3 25 45.9
Sept. 23,	-1.3_4		-1.4_3			-1.3	55.2	3 48 56.5
Oct. 28,	-0.6_5		-1.0_3			-0.8	34.7	4 8 35.5
Dec. 12,	-0.3_5					-0.3	14.2	4 15 14.5
1859, Aug. 21,	-2.9_4		-3.0_2			-2.9	26.2	-2 35 29.1
Sept. 23,	-2.7_5		-2.9_3			-2.8	45.6	2 56 48.4
Nov. 8,	-2.1_5		-2.3_3			-2.2	19.5	3 21 21.7
Dec. 14,	-1.0_4		-2.6_2			-1.5	48.4	3 24 49.9
1860, Aug. 20,	-4.7_2					-4.7	13.8	-1 43 18.5
Sept. 23,	-3.6_5		-4.2_2			-3.8	3.1	2 5 6.9
Oct. 31,	-3.1_5		-4.0_3			-3.4	0.3	2 27 3.7
Dec. 13,	-4.4_1					-4.4	24.1	2 34 28.5
1861, Aug. 22,	-5.2_2					-5.2	4.6	-0 52 9.8
Sept. 18,	-5.4_5		-6.1_2		-5.8_2	-5.6	10.8	1 9 16.4
Oct. 30,	-4.9_4		-6.1_5	-4.1_2	-5.3_2	-5.1	35.1	1 34 40.2
Dec. 7,	-4.9_3		-6.1_2	-4.6_2	-5.3_2	-5.2	59.8	-1 44 5.0
1862, Aug. 24,	-6.6_3					-6.6	53.0	-0 0 59.6
Sept. 23,	-6.1_5				-7.5_0	-6.1	56.3	0 20 2.4
Nov. 6,	-6.3_4			-5.7_1	-8.4_0	-6.1	38.0	0 45 44.1
Dec. 15,	-6.4_2			-6.7_1	-7.6_0	-6.5	45.8	-0 52 52.3
1863, Aug. 28,	-9.6_1					-9.6	11.6	$+0$ 49 2.0
Sept. 27,	-8.4_3			-8.9_2	-8.4_3	-8.5	61.5	0 29 53.0
Nov. 17,	-8.1_2			-8.7_4	-9.3_4	-8.8	13.6	$+0$ 2 4.8
Dec. 12,	-8.1_1			-8.2_1	-8.8_3	-8.6	53.1	-0 2 1.7
1864, Aug. 7,				-10.7_2		-10.7	53.8	$+1$ 50 43.1
Oct. 1,	-10.8_1				-10.7_3	-10.7	19.0	1 19 8.3
Nov. 12,	-10.2_3			-10.5_4	-9.4_3	-10.1	38.7	0 55 28.6
Dec. 17,				-9.9_1	-9.5_2	-9.7	21.3	$+0$ 49 11.6

REMARKS ON THE PRECEDING TABLE.

The processes to which we have subjected the observations ought, it would seem, to eliminate every source of constant differences between those made at different observatories. But there are still two well-marked cases of systematic differences in the right ascensions, namely, in the Cambridge observations of the first five years, and the Albany observations of the last four. The differences between the corrections finally concluded from all the observations, and those concluded from Cambridge and Albany, are, it will be seen, as follows:

Date.	Conc.—Camb.	Date.	Conc.—Albany.
	s		s
1846, Oct.	+ 0.02	1861, Sept.	— 0.09
Nov.	+ 0.06	Oct.	— 0.06
1847, July,	+ 0.02	Dec.	— 0.05
Aug.	+ 0.03	1862, Aug.	— 0.06
Oct.	+ 0.08	Sept.	— 0.08
Nov.	+ 0.07	Nov.	— 0.07
1848, July,	+ 0.07	Dec.	— 0.09
Aug.	+ 0.07	1863, Sept.	— 0.06
Oct.	+ 0.01	Nov.	— 0.04
Nov.	+ 0.07	Dec.	— 0.01
1849, Sept.	+ 0.06	1864, Oct.	— 0.05
Oct.	+ 0.10	Nov.	— 0.08
Nov.	— 0.03	Dec.	— 0.03
1850, Aug.	+ 0.04		
Oct.	+ 0.05		
Nov.	+ 0.01		

The constancy of signs here exhibited can hardly be attributed to chance in the case of Cambridge, and not at all in the case of Albany. The only cause to which I can attribute it is a habit of registering the transit of Neptune earlier or later than that of a bright star. Such a habit would seem to pertain to the observer rather than the instrument, and, therefore, less to be feared as the number of observers is increased. On account of its possible existence, the weights of the results of any one observatory have not been supposed proportional to the number of observations, but each has been subject to a constant probable error of at least $0^s.02$ when observations were made by eye and ear, and $0^s.01$ when made with chronograph, however great the number of observations.

Albany exhibits the anomaly that the real systematic error seems greater than the probable accidental error. The latter is of the smallest class, as might be anticipated from the facts that the observations are made with a first-class instrument, in a good atmosphere, and are recorded with the electro-chronograph. They have, therefore, been treated in such a way that, while they should enter the absolute longitudes with a very small weight, they should enter the relative longitudes at different times of the year, in other words, the radius vector, with

as much weight as those of any other observatory. This has been effected by applying the constant correction —$0^s.04$ to all the results before combining them.

Anomalies somewhat similar are exhibited by the Paris declinations from 1860 to 1861, and by the Washington declinations of 1861. In the case of Washington, they may be accounted for by the circumstance that the systematic corrections for 1861 depend mainly on observations made in 1863, very few declinations of fundamental stars being observed in 1861–62. But it does not seem so easy to account for the discrepancy between the Paris and Greenwich results. A comparison of them shows that while the Paris observations systematically place the ten fundamental stars adopted as our standard about $0''.8$ farther north than Greenwich, their positions of Neptune, and of some small stars near the equator, substantially agree.

§ 28. The preceding normal right ascensions and declinations are next converted into apparent ecliptic longitudes and latitudes, for the purpose of comparison with the provisional theory. For this purpose Hansen's obliquity of the ecliptic has been adopted, so as to agree with the motion of the ecliptic adopted in the preceding chapter. In the following table we give for each date—1. The longitude from observation, obtained as just stated. 2. The seconds of longitude from provisional theory, as given on p. 43. 3. The excess of the theoretical over the observed longitude. 4, 5, 6. The corresponding quantities relative to the latitude.

Geocentric Apparent Longitudes and Latitudes of Neptune derived from Observation.

Date.	Longitude. Observation.	Longitude. Theory.	Error of Theory.	Latitude. Observation.	Latitude. Theory.	Error of Theory.
	° ′ ″	″	″			
1795, May 9,	214 37 20.4	19.1	—1.3	+1 50 33.3	34.4	+1.1
1846, Oct. 14,	325 31 35.0	34.9	—0.1	—0 31 55.8	56.0	—0.2
Nov. 14,	325 23 24.2	23.2	—1.0	0 31 43.5	44.0	—0.5
1847, July 26,	329 41 22.9	22.0	—0.9	0 35 24.1	25.9	—1.8
Aug. 17,	329 7 18.9	18.3	—0.6	0 35 45.7	47.7	—2.0
Oct. 8,	327 52 10.4	10.4	0.0	0 35 57.6	58.8	—1.2
Nov. 18,	327 36 56.9	56.3	—0.6	0 35 38.3	38.6	—0.3
1848, July 25,	331 59 6.7	5.0	—1.7	0 39 21.8	22.8	—1.0
Aug. 29,	331 3 16.6	15.8	—0.8	0 39 54.0	55.2	—1.2
Oct. 6,	330 8 55.3	55.3	0.0	0 39 56.8	57.8	—1.0
Nov. 17,	329 59 22.8	22.4	—0.4	0 39 32.3	32.8	—0.5
1849, Sept. 1,	333 15 38.7	38.6	—0.1	0 43 51.8	52.7	—0.9
Oct. 15,	332 15 32.6	32.4	—0.2	0 43 48.6	49.2	—0.6
Nov. 25,	332 4 17.1	16.7	—0.4	0 43 15.7	17.1	—1.4
1850, Aug. 28,	335 39 38.5	38.5	0.0	0 47 41.9	42.7	—0.8
Oct. 15,	334 31 10.3	9.5	—0.8	0 47 39.9	41.1	—1.2
Nov. 20,	334 15 24.4	23.8	—0.6	0 47 8.6	9.5	—0.9
1851, Sept. 2,	337 48 58.5	58.1	—0.4	0 51 32.6	33.7	—1.1
Oct. 14,	336 48 12.7	10.9	—1.8	0 51 30.0	30.0	0.0
Nov. 20,	336 28 32.5	31.9	—0.6	0 50 53.3	54.1	—0.8
1852, Aug. 7,	340 46 10.7	11.0	+0.3	0 54 49.8	51.6	—1.8
Sept. 5,	340 0 11.1	10.3	—0.8	0 55 18.8	19.5	—0.7
Oct. 12,	339 5 43.2	43.0	—0.2	0 55 14.6	14.8	—0.2
Nov. 28,	338 43 24.0	23.4	—0.6	0 54 23.4	23.3	+0.1
1853, Sept. 1,	342 24 48.0	47.7	—0.3	0 58 54.1	55.9	—1.8
Oct. 15,	341 19 0.8	0.2	—0.6	0 58 52.0	52.4	—0.4
Nov. 24,	340 56 4.3	3.0	—1.3	0 58 4.5	4.7	—0.2
1854, Aug. 30,	344 46 18.1	17.8	—0.3	1 2 25.8	27.5	—1.7
Sept. 24,	344 5 33.4	33.2	—0.2	1 2 35.8	37.0	—1.2
Oct. 27,	343 23 17.5	17.3	—0.2	1 2 14.4	15.3	—0.9
Dec. 5,	343 12 3.1	2.8	—0.3	1 1 19.4	19.1	+0.3
1855, Aug. 10,	347 34 55.1	54.2	—0.9	1 5 26.8	28.0	—1.2
Sept. 8,	346 49 57.9	57.0	—0.9	1 6 1.4	1.8	—0.4
Oct. 22,	345 45 7.4	6.2	—1.2	1 5 50.2	50.1	+0.1
Nov. 29,	345 24 25.6	25.3	—0.3	1 4 53.7	54.8	—1.1

GEOCENTRIC APPARENT LONGITUDES AND LATITUDES OF NEPTUNE DERIVED FROM OBSERVATION (Cont.).

Date.	Longitude. Observation.	Longitude. Theory.	Error of Theory.	Latitude. Observation.	Latitude. Theory.	Error of Theory.
	° ′ ″	″	″			
1856, Aug. 8,	349 54 4.2	3.3	—0.9	—1 8 43.7	44.8	—1.1
Sept. 13,	348 58 37.9	37.4	—0.5	1 9 26.5	27.2	—0.7
Oct. 26,	347 56 49.5	48.8	—0.7	1 9 7.1	8.2	—1.1
Nov. 17,	347 40 53.7	53.1	—0.6	1 8 34.0	35.3	—1.3
1857, Aug. 13,	352 5 16.8	16.5	—0.3	1 12 5.6	6.2	—0.6
Sept. 21,	351 4 3.1	2.0	—1.1	1 12 45.6	46.1	—0.5
Oct. 24,	350 16 10.7	9.6	—1.1	1 12 28.3	28.3	0.0
Dec. 8,	349 54 2.4	1.4	—1.0	1 11 11.8	11.8	0.0
1858, Aug. 18,	354 16 21.7	20.9	—0.8	1 15 19.7	21.0	—1.3
Sept. 23,	353 19 17.3	17.2	—0.1	1 15 56.1	56.6	—0.5
Oct. 28,	352 28 49.3	49.2	—0.1	1 15 34.2	34.8	—0.6
Dec. 12,	352 8 48.3	46.7	—1.6	1 14 11.6	11.8	—0.2
1859, Aug. 21,	356 30 3.7	2.9	—0.8	1 18 25.2	25.8	—0.6
Sept. 23,	355 37 45.0	44.6	—0.4	1 18 59.5	60.0	—0.5
Nov. 8,	354 34 30.3	29.4	—0.9	1 18 22.8	23.0	—0.2
Dec. 14,	354 22 53.5	52.5	—1.0	1 17 8.5	9.3	—0.8
1860, Aug. 20,	358 47 47.9	48.1	+0.2	1 21 17.0	17.7	—0.7
Sept. 23,	357 54 28.9	28.1	—0.8	1 21 55.4	56.4	—1.0
Oct. 31,	356 58 23.5	22.4	—1.1	1 21 31.1	32.2	—1.1
Dec. 13,	356 36 25.2	24.7	—0.5	1 20 4.5	4.6	—0.1
1861, Aug. 22,	1 2 43.6	42.4	—1.2	1 24 4.8	6.0	—1.2
Sept. 18,	0 21 9.6	7.6	—2.0	1 24 41.8	42.2	—0.4
Oct. 30,	359 16 40.6	38.4	—2.2	1 24 24.0	24.6	—0.6
Dec. 7,	358 50 4.3	3.3	—1.0	1 23 7.0	7.9	—0.9
1862, Aug. 24,	3 17 37.0	34.7	—2.3	1 26 46.0	46.7	—0.7
Sept. 23,	2 31 9.8	7.7	—2.1	1 27 24.2	25.7	—1.5
Nov. 6,	1 25 28.0	26.1	—1.9	1 26 56.1	57.3	—1.2
Dec. 15,	1 4 11.5	10.0	—1.5	1 25 29.0	30.0	—1.0
1863, Aug. 28,	5 29 46.2	46.2	0.0	1 29 22.8	23.7	—0.9
Sept. 27,	4 42 51.8	49.6	—2.2	1 29 59.4	60.3	—0.9
Nov. 17,	3 30 59.6	58.0	—1.6	1 29 12.1	12.5	—0.4
Dec. 12,	3 18 20.4	18.2	—2.2	1 28 12.1	12.4	—0.3
1864, Aug. 7,	8 9 22.8	20.6	—2.2	1 30 52.9	53.7	—0.8
Oct. 1,	6 52 59.8	56.7	—3.1	1 32 26.8	27.0	—0.2
Nov. 12,	5 51 40.4	37.8	—2.6	1 31 48.0	49.1	—1.1
Dec. 17,	5 32 30.3	27.7	—2.6	—1 30 22.6	23.1	—0.5

CHAPTER IV.

RESULTS OF THE COMPARISON OF THE THEORETICAL WITH THE OBSERVED POSITIONS OF NEPTUNE.

§ 29. The first question of the present chapter will be whether the observations of Neptune can be satisfied within the limits of their probable errors by suitable changes in the elements of the orbit of Neptune and the masses of the disturbing planets.

No admissible change in the mass either of Jupiter or Saturn will sensibly affect the perturbations of Neptune. The mass of Uranus will, therefore, be the only one the correction of which need be taken into account.

The errors of the provisional latitude of Neptune are so small that the errors of the longitude in orbit may be taken as sensibly the same with the errors of ecliptic longitude. The latter give equations of condition between the following unknown quantities.

Correction of the mean longitude of Neptune.
" " mean motion of Neptune.
" " eccentricity $\times$ sin. perihelion of Neptune.
" " eccentricity $\times$ cos. perihelion of Neptune.
" " mass of Uranus.

But if we attempt to solve by least squares the equations between these corrections, we shall be met with the difficulty set forth in the introduction, and our normal equations will be equivalent to only three, unless we include a great number of decimals in the computation. We shall, therefore, make a linear transformation of the unknown quantities, on the principles already referred to, and suggested by the following considerations.

The true longitude of Neptune has been less than its mean longitude, and its true motion has been greater than its mean motion, ever since its optical discovery. From these circumstances the difficulty in question arises. We may obviate it by substituting for the mean longitude and mean motion of Neptune during an entire revolution its average longitude and heliocentric motion during the period of the modern observations. Suppose an imaginary planet to move uniformly in the orbit of Neptune in such a way that its average longitude and motion have been the same as the average longitude and motion of Neptune during the last nineteen years, and let x be its longitude, 1850, Jan. 0, and x' its annual motion. We may then make the eccentricity and perihelion of Neptune to depend analytically upon the deviation of its motion from that of the hypothetical planet, as it must depend really, *because this deviation is the only real datum which we possess to reason from,* the Lalande observations excepted. It is to be remarked

that both the longitude and motion of the hypothetical planet are entirely arbitrary.

For the differential coefficients of the elements with respect to the heliocentric co-ordinates, we have

$$\frac{dv}{d\varepsilon} = 1 + 2\,k\cos l + 2\,h\sin l.$$

$$\frac{dv}{dn} = t\,.\,\frac{dv}{d\varepsilon}.$$

$$\frac{dv}{dh} = -2\cos l - \tfrac{5}{2}\,h\sin 2\,l - \tfrac{5}{2}\,k\cos 2\,l.$$

$$\frac{dv}{dk} = \quad 2\sin l + \tfrac{5}{2}\,k\sin 2\,l - \tfrac{5}{2}\,h\cos 2\,l.$$

$$\frac{1}{a}\,\frac{dr}{d\varepsilon} = k\sin l - h\cos l.$$

$$\frac{1}{a}\,\frac{dr}{dn} = -\frac{2r}{3an} + \frac{t}{a}\,.\,\frac{dr}{d\varepsilon}$$

$$\frac{1}{a}\,\frac{dr}{dh} = -\sin l + h - k\sin 2\,l + h\cos 2\,l.$$

$$\frac{1}{a}\,\frac{dr}{dk} = -\cos l + k - h\sin 2\,l - k\cos 2\,l.$$

In accordance with what has been proposed, we shall substitute for ε and n the quantities x and x', connected with them by the relations

$$\begin{aligned} x &= \varepsilon + \alpha h + \beta k \\ x' &= n + \alpha' h + \beta' k \end{aligned} \qquad (1)$$

α and β being approximately the average values of $-2\cos l$ and $+2\sin l$ during the last nineteen years, and α' and β' the average values of $2\,n\sin l$ and $2\,n\cos l$ during the same time. We shall take

$$\begin{aligned} \alpha &= -1.77 \qquad & \alpha' &= -0.018 \\ \beta &= -0.85 \qquad & \beta' &= +0.073. \end{aligned} \qquad (2)$$

Then, considering v as a function of x, y, h, and k, and enclosing the new differential coefficients in parentheses, we have, by suitable transformations,

$$\left(\frac{dv}{dx}\right) = \frac{dv}{d\varepsilon};\quad \left(\frac{dv}{dx'}\right) = \frac{dv}{dn};\quad \left(\frac{dr}{dx}\right) = \frac{dr}{d\varepsilon};\quad \left(\frac{dr}{dx'}\right) = \frac{dr}{dn}$$

$$\left(\frac{dv}{dh}\right) = \frac{dv}{dh} - (\alpha + \alpha' t)\,\frac{dv}{d\varepsilon}$$

$$\left(\frac{dv}{dk}\right) = \frac{dv}{dk} - (\beta + \beta' t)\,\frac{dv}{d\varepsilon} \qquad (3)$$

$$\frac{1}{a}\left(\frac{dr}{dh}\right) = \frac{1}{a}\,\frac{dr}{dh} - (\alpha + \alpha' t)\cdot\frac{1}{a}\,\frac{dr}{d\varepsilon} + \tfrac{2}{3}\,\alpha'\cdot\frac{r}{an}$$

$$\frac{1}{a}\left(\frac{dr}{dk}\right) = \frac{1}{a}\,\frac{dr}{dk} - (\beta + \beta' t)\cdot\frac{1}{a}\,\frac{dr}{d\varepsilon} + \tfrac{2}{3}\,\beta'\cdot\frac{r}{an}$$

Putting λ for the geocentric longitude, and Δ for the distance from the earth, the differential coefficients of the geocentric with respect to the heliocentric co-ordinates will be

$$\begin{aligned} \frac{d\lambda}{dv} &= \frac{r}{\Delta}\cos(v-\lambda), \\ a\frac{d\lambda}{dr} &= \frac{a}{\Delta}\sin(v-\lambda); \end{aligned} \tag{4}$$

and the coefficients of the equations of conditions will be

$$\begin{aligned} \frac{d\lambda}{dx} &= \frac{d\lambda}{dv}\ \frac{dv}{d\varepsilon} + a\frac{d\lambda}{dr}\ \frac{1}{a}\ \frac{dr}{d\varepsilon} \\ \frac{d\lambda}{dx'} &= \frac{d\lambda}{dv}\ \frac{dv}{dn} + a\frac{d\lambda}{dr}\ \frac{1}{a}\ \frac{dr}{dn} \\ \left(\frac{d\lambda}{dh}\right) &= \frac{d\lambda}{dv}\left(\frac{dv}{dh}\right) + a\frac{d\lambda}{dr}\ \frac{1}{a}\left(\frac{dr}{dh}\right) \\ \left(\frac{d\lambda}{dk}\right) &= \frac{d\lambda}{dv}\left(\frac{dv}{dk}\right) + a\frac{d\lambda}{dr}\ \frac{1}{a}\left(\frac{dr}{dk}\right) \end{aligned} \tag{5}$$

The perturbations in the geocentric longitude of Neptune produced by Uranus will be—

1. Perturbations of the true heliocentric longitude multiplied by $\frac{d\lambda}{dv}$;

2. Perturbations of radius vector multiplied by $\frac{d\lambda}{dr}$, for which has been taken $\delta \log r \times \frac{a}{M}\ \frac{d\lambda}{dr}$

Of course the effect of the long-period and secular perturbations of the elements produced by the action of Uranus must be included in the perturbations of Neptune.

Representing by μ the factor by which the assumed mass of Uranus must be multiplied, so that the true mass shall be

$$\frac{1+\mu}{21000},$$

the computed perturbations produced by Uranus will be the coefficients of μ in the equations of condition.

§ 30. The residuals in longitude thus give the following equations between the unknown quantities, which are numbered in the order of time, but grouped somewhat differently.

No.	Date.	Equation.						P.	M.
							"		
1	1795, May 9,	$0=1.02\delta x$	$-55.7\delta x'$	$+2.454\delta h$	$+3.742\delta k$	$+34.6\mu$	−1.3	½	¼
4	1847, July 26,	1.02	−2.2	+0.009	−0.001	+1.17	−0.9	4	2
5	1847, Aug. 17,	1.04	−2.4	+0.010	+0.003	+1.34	−0.6	10	5
2	1846, Oct. 14,	1.02	−3.8	+0.035	+0.024	+1.79	−0.1	6	2
6	47, Oct. 8,	1.03	−2.7	+0.012	+0.014	+1.66	0.0	8	3
3	1846, Nov. 14,	1.01	−3.7	+0.034	+0.025	+1.82	−1.0	7	2
7	47, Nov. 18,	1.01	−2.7	+0.011	+0.017	+1.73	−0.6	7	2
8	1848, July 25,	1.04	−1.2	−0.011	−0.008	+1.16	−1.7	5	2
9	1848, Aug. 29,	1.04	−1.4	−0.010	0.000	+1.39	−0.8	8	3
12	49, Sept. 1,	1.04	−0.4	−0.027	−0.006	+1.47	−0.1	6	2
15	50, Aug. 28,	1.04	+0.7	−0.042	−0.011	+1.55	0.0	5	2
10	1848, Oct. 6,	1.03	−1.7	−0.008	+0.007	+1.61	0.0	8	3
13	49, Oct. 15,	1.03	−0.7	−0.026	+0.002	+1.69	−0.2	8	3
16	50, Oct. 15,	1.03	+0.4	−0.041	−0.002	+1.79	−0.8	7	2
11	1848, Nov. 17,	1.01	−1.7	−0.009	+0.010	+1.68	−0.4	4	1
14	49, Nov. 25,	1.01	−0.7	−0.026	+0.005	+1.74	−0.4	6	2
17	50, Nov. 20,	1.01	+0.4	−0.041	+0.001	+1.85	−0.6	7	2
21	1852, Aug. 7,	1.04	+3.0	−0.062	−0.020	+1.85	+0.3	3	1
18	1851, Sept. 2,	1.04	+1.7	−0.054	−0.013	+1.76	−0.4	6	2
22	52, Sept. 5,	1.04	+2.8	−0.063	−0.014	+2.00	−0.8	5	2
25	53, Sept. 1,	1.04	+3.9	−0.069	−0.016	+2.31	−0.3	5	2
19	1851, Oct. 14,	1.03	+1.4	−0.054	−0.005	+1.95	−1.8	4	1
23	52, Oct. 12,	1.03	+2.5	−0.063	−0.008	+2.16	−0.2	5	2
26	53, Oct. 15,	1.04	+3.5	−0.070	−0.008	+2.44	−0.6	6	2
20	1851, Nov. 20,	1.01	+1.3	−0.053	−0.003	+2.00	−0.6	4	1
24	52, Nov. 28,	1.01	+2.4	−0.062	−0.004	+2.22	−0.6	4	1
27	53, Nov. 24,	1.01	+3.4	−0.069	−0.004	+2.53	−1.3	5	2
28	1854, Aug. 30,	1.04	+5.0	−0.072	−0.016	+2.62	−0.3	6	2
32	55, Aug. 10,	1.04	+6.1	−0.071	−0.018	+2.89	−0.9	8	3
36	56, Aug. 8,	1.04	+7.2	−0.067	−0.017	+3.31	−0.9	8	3
29	1854, Sept. 24,	1.04	+4.8	−0.073	−0.012	+2.75	−0.2	4	1
33	55, Sept. 8,	1.04	+6.0	−0.072	−0.014	+3.11	−0.9	9	3
37	56, Sept. 13,	1.05	+7.0	−0.070	−0.011	+3.44	−0.5	9	3
30	1854, Oct. 27,	1.03	+4.5	−0.073	−0.006	+2.84	−0.2	6	2
34	55, Oct. 22,	1.04	+5.6	−0.074	−0.006	+3.14	−1.2	7	2
38	56, Oct. 26,	1.03	+6.6	−0.072	−0.004	+3.57	−0.7	9	3
31	1854, Dec. 5,	1.01	+4.4	−0.072	−0.004	+2.85	−0.3	4	1
35	55, Nov. 29,	1.02	+5.4	−0.074	−0.003	+3.17	−0.3	8	3
39	56, Nov. 17,	1.02	+6.5	−0.072	−0.002	+3.59	−0.6	8	3
40	1857, Aug. 13,	1.04	+8.2	−0.061	−0.014	+3.83	−0.3	7	2
44	58, Aug. 18,	1.04	+9.3	−0.052	−0.011	+4.39	−0.8	9	3
48	59, Aug. 21,	1.04	+10.3	−0.040	−0.007	+4.94	−0.8	9	3
41	1857, Sept. 21,	1.04	+8.0	−0.065	−0.007	+3.97	−1.1	9	3
45	58, Sept. 23,	1.05	+9.1	−0.056	−0.005	+4.51	−0.1	9	3
49	59, Sept. 23,	1.05	+10.1	−0.044	−0.002	+5.05	−0.4	10	3
42	1857, Oct. 24,	1.04	+7.7	−0.067	−0.002	+4.05	−1.1	6	2
46	58, Oct. 28,	1.04	+8.8	−0.059	+0.001	+4.57	−0.1	10	3
50	59, Nov. 8,	1.03	+9.7	−0.047	+0.005	+5.11	−0.9	9	3
43	1857, Dec. 8,	1.01	+7.5	−0.066	+0.001	+4.09	−1.0	9	3
47	58, Dec. 12,	1.01	+8.5	−0.058	+0.004	+4.55	−1.6	8	3
51	59, Dec. 14,	1.02	+9.5	−0.047	+0.007	+5.09	−1.0	8	3
52	1860, Aug. 20,	1.04	+11.4	−0.024	−0.004	+5.56	+0.2	6	2
56	61, Aug. 22,	1.04	+12.5	−0.005	−0.001	+6.16	−1.2	6	2
60	62, Aug. 24,	1.04	+13.5	+0.016	+0.003	+6.79	−2.3	9	3

No.	Date.	Equation.						P.	M.
						$''$	$''$		
53	1860, Sept. 23,	$0=1.05\delta x$	$+11.2\delta x'$	$-0.028\delta h$	$+0.002\delta k$	$+5.67\mu$	-0.8	10	3
57	61, Sept. 18,	1.05	+12.3	−0.009	+0.004	+6.26	−2.0	11	3
61	62, Sept. 23,	1.05	+13.4	+0.012	+0.008	+6.88	−2.1	11	3
54	1860, Oct. 31,	1.04	+10.8	−0.033	+0.008	+5.71	−1.1	9	3
58	61, Oct. 30,	1.04	+11.9	−0.015	+0.011	+6.31	−2.2	10	3
62	62, Nov. 6,	1.04	+12.9	+0.005	+0.015	+6.89	−1.9	11	3
55	1860, Dec. 13,	1.02	+10.5	−0.033	+0.011	+5.68	−0.5	4	1
59	61, Dec. 7,	1.02	+11.6	−0.016	+0.014	+6.25	−1.0	11	3
63	62, Dec. 15,	1.02	+12.6	+0.004	+0.018	+6.81	−1.5	10	3
64	1863, Aug. 28,	1.04	+14.6	+0.040	+0.006	+7.42	0.0	4	1
68	64, Aug. 7,	1.04	+15.6	+0.070	+0.007	+7.94	−2.2	5	2
65	1863, Sept. 27,	1.05	+14.4	+0.036	+0.012	+7.50	−2.2	10	4
69	64, Oct. 1,	1.05	+15.4	+0.063	+0.015	+8.16	−3.1	8	3
66	1863, Nov. 17,	1.04	+13.9	+0.028	+0.019	+7.49	−1.6	9	3
70	64, Nov. 12,	1.04	+15.0	+0.054	+0.022	+8.18	−2.6	10	4
67	1863, Dec. 12,	1.02	+13.7	+0.027	+0.021	+7.45	−2.2	9	3
71	64, Dec. 17,	1.02	+14.7	+0.052	+0.024	+8.13	−2.6	8	3

In order to lessen the labor of solving these equations, they have been divided into groups, with respect to the years of observation, and the difference of heliocentric longitude of the earth and planet. The nineteen years of modern observations have been divided into seven groups, of which the first and last each include two years, and each of the intermediate ones three years. Then, in each group of years, the equations which pertain to corresponding times of the year are grouped together, and will be combined into one.

The numbers in column P. are assumed as the "measure of precision" of the residuals of each equation. These numbers were inferred from the numbers and excellence of the observations on which each normal was founded, the unit of precision was assumed to correspond to the probable error $1''.5$, and no equation was allowed to have a precision exceeding 11. Hence the assumed probable error of each equation is $\frac{1''.5}{P}$. But the residuals left after the final solution show that the measures of precision attached to the modern positions are too great, and that their probable errors are really about $\frac{2''.4}{P}$.

Column M. gives the number by which the individual equations must be multiplied in order that when those of each group are added together, the precision of their sum may be 2. It is approximately $\frac{P}{2\sqrt{n}}$, n being the number of individual equations in the group.

To make the solution more convenient with respect to decimals, the coefficients of $\delta x'$ will all be multiplied by 10, and those of δh and δk divided by 10, after condensing the equations in the manner proposed.

Thus the following twenty-nine homogeneous equations are obtained:

				″	″	
$0 = 0.25\delta x$	$-\ 1.39 \times 10x'$	$+\ 6.14\frac{\delta h}{10}$	$+\ 9.36\frac{\delta k}{10}$	$+\ 8.6\mu$	$-\ 0.3$	
2.04	− 0.44	+ 0.18	− 0.02	+ 2.3	− 1.8	
5.20	− 1.20	+ 0.50	+ 0.15	+ 6.7	− 3.0	
5.13	− 1.57	+ 1.06	+ 0.90	+ 8.6	− 0.2	
4.04	− 1.28	+ 0.90	+ 0.84	+ 7.1	− 3.2	
2.08	− 0.24	− 0.22	− 0.16	+ 2.3	− 3.4	
7.28	− 0.36	− 1.68	− 0.34	+ 10.2	− 2.6	
8.24	− 0.64	− 1.84	+ 0.23	+ 13.5	− 2.2	
5.05	− 0.23	− 1.43	+ 0.22	+ 8.9	− 2.4	
1.04	+ 0.30	− 0.62	− 0.20	+ 1.8	+ 0.3	
6.24	+ 1.68	− 3.72	− 0.86	+ 12.1	− 3.0	
5.16	+ 1.34	− 3.20	− 0.37	+ 11.2	− 3.4	
4.04	+ 1.05	− 2.53	− 0.15	+ 9.3	− 3.8	
8.33	+ 4.99	− 5.58	− 1.37	+ 23.8	− 6.0	
7.31	+ 4.38	− 4.99	− 0.87	+ 22.1	− 4.4	(7)
7.23	+ 4.00	− 5.10	− 0.36	+ 22.7	− 4.9	
7.13	+ 4.01	− 5.10	− 0.19	+ 23.1	− 3.0	
8.32	+ 7.52	− 3.98	− 0.82	+ 35.7	− 5.4	
9.42	+ 8.16	− 4.95	− 0.42	+ 40.6	− 4.8	
8.29	+ 7.09	− 4.52	+ 0.14	+ 37.1	− 5.2	
9.10	+ 7.65	− 5.13	+ 0.36	+ 41.2	− 10.8	
7.28	+ 8.83	− 0.10	− 0.01	+ 43.8	− 8.9	
9.44	+ 11.07	− 0.75	+ 0.42	+ 56.4	− 14.7	
9.36	+ 10.68	− 1.29	+ 1.02	+ 56.7	− 15.6	
7.14	+ 8.31	− 0.69	+ 1.07	+ 44.9	− 8.0	
3.11	+ 4.58	+ 1.80	+ 0.20	+ 23.3	− 4.4	
7.35	+ 10.38	+ 3.33	+ 0.93	+ 54.5	− 18.1	
7.27	+ 10.17	+ 3.07	+ 1.45	+ 55.2	− 15.2	
6.12	+ 8.52	+ 2.37	+ 1.35	+ 46.7	− 14.4	

§ 31. Treating these equations by the method of least squares, but leaving μ indeterminate for the present, we have the four normals

				″	″	
$1277.71x$	$+\ 935.29(10x')$	$-\ 350.59 \cdot \frac{\delta h}{10}$	$+\ 22.46\frac{\delta k}{10}$	$+\ 5431.7\mu$	$-\ 1263.39 = 0$	
935.29	+ 1010.58	− 190.60	+ 24.58	+ 5178.1	− 1240.38	(8)
− 350.59	− 190.60	+ 305.88	+ 90.30	− 935.5	+ 146.81	
22.46	+ 24.58	+ 90.30	+ 101.33	+ 317.9	− 74.89	

The solution of these equations gives the following values of the unknown quantities in terms of μ.

$$
\begin{aligned}
\delta x &= +\ 0.650 &&-\ 2.067\mu \\
\delta x' &= +\ 0.0800 &&-\ 0.342\mu \\
\delta h &= +\ 8.76 &&-12.18\mu \\
\delta k &= -3.79 &&-\ 7.64\mu
\end{aligned}
\qquad (9)
$$

(The first two constant terms and μ-coefficients are marked ″.)

Substituting these values of the corrections in equations (7), we have the following residuals, which are grouped, as before, according to the time of year of the normals on which the equations were founded. Thus, the first residual of each series of modern observations corresponds to positions of Neptune observed when the planet culminated after $13^h\ 30^m$ during the years to which the series belongs.

	h. m.		h. m.
The second, to observations between	10 30	and	13 30
The third, to observations between	7 30	and	10 30
The fourth, before	7 30		

We first give the residuals from the equations (7), each of which is supposed to be of equal precision; then the numbers by which the errors of observation are multiplied to reduce them to the assumed standard of precision derived from (6), column M.; and, finally, the apparent errors of the theory derived from observations themselves, formed by dividing the residuals of the equations by the measures of precision.

	Residuals of equations.			Actual mean residuals or apparent errors of theory.	
	"			"	"
1st series, 1795,	+ 0.58	— 1.8μ	$\frac{1}{4}$	+ 2.3	— 7.2μ
2d series, 1846–1847,	— 0.7	— 0.6μ	2	— 0.35	— 0.30μ
	— 0.2	— 0.8μ	5	— 0.04	— 0.16μ
	+ 2.4	+ 1.5μ	5	+ 0.48	+ 0.30μ
	— 1.1	+ 1.4μ	4	— 0.28	+ 0.35μ
3d series, 1848–1850,	— 2.3	— 0.7μ	2	— 1.15	— 0.35μ
	+ 0.5	— 1.5μ	7	+ 0.07	— 0.21μ
	+ 1.0	+ 0.6μ	8	+ 0.12	+ 0.08μ
	— 0.7	+ 0.8μ	5	— 0.14	+ 0.16μ
4th series, 1851–1853,	+ 0.8	— 0.4μ	1	+ 0.80	— 0.40μ
	— 0.6	— 0.5μ	6	— 0.10	— 0.08μ
	— 1.6		5	— 0.32	
	— 2.5	+ 0.6μ	4	— 0.62	+ 0.15μ
5th series, 1854–1856,	— 1.0	— 2.7μ	8	— 0.12	— 0.34μ
	— 0.2	— 1.2μ	7	— 0.03	— 0.17μ
	— 1.4	+ 0.4μ	7	— 0.20	+ 0.06μ
	+ 0.4	+ 1.0μ	7	+ 0.06	+ 0.14μ
6th series, 1857–1859,	+ 2.8	— 1.7μ	8	+ 0.35	— 0.21μ
	+ 3.7	— 0.4μ	9	+ 0.41	— 0.05μ
	+ 1.8	+ 0.9μ	8	+ 0,22	+ 0.11μ
	— 3.4	+ 2.2μ	9	— 0.38	+ 0.24μ
7th series, 1860–1862,	+ 2.8	— 1.4μ	7	+ 0.40	— 0.20μ
	— 0.6	— 0.5μ	9	— 0.07	— 0.06μ
	— 2.5	+ 1.6μ	9	— 0.28	+ 0.18μ
	+ 2.2	+ 1.7μ	7	+ 0.31	+ 0.24μ

	Residuals of equations.			Actual mean residuals or apparent errors of theory.	
	$''$	$''$		$''$	$''$
8th series, 1863–1864,	$+2.8$	-1.2μ	3	$+0.93$	-0.40μ
	-2.4	-0.9μ	7	-0.34	-0.13μ
	-0.3	$+0.7\mu$	7	-0.04	$+0.10\mu$
	-2.0	$+0.9\mu$	6	-0.33	$+0.15\mu$

§ 32. The coefficients of μ, taken negatively, represent the changes which would be produced in the residuals if we suppose the mass of Uranus to be nothing. It will be seen that these coefficients are generally smaller than the residuals themselves, and that their actual effect on the modern residuals never amounts to more than four-tenths of a second. Supposing that the modern observations cannot be relied on within this limit of error, we should arrive at this remarkable result,—that if the planet Uranus were unknown, its existence could scarcely be inferred from all the observations hitherto made on Neptune, unless these were combined in such a way as to show the systematic error of the theoretical radius vector. In fact, the orbit of Neptune, computed without regard to the perturbations of Uranus, would only exhibit an error of $9''$ when compared with Lalande's position; and a discussion of the modern observations would exhibit no sensible error in the heliocentric longitudes. This circumstance furnishes a very good illustration of the propriety of developing the long-period perturbations, the coefficients of which amount to whole minutes, as perturbations of the elements which shall vanish at the epoch 1850.

Under these circumstances, no reliable correction of the mass of Uranus can be concluded from the motions of Neptune. The solution of the preceding residuals does, indeed, indicate an increase of this mass by one-third, which seems altogether inadmissible, and is certainly very unreliable. Of the twenty-nine residuals, fifteen indicate an increase of the mass, thirteen a diminution, and for one the coefficient of μ vanishes: so that the increase of the mass of Uranus is indicated only by the fact that the residuals which favor it are generally a little larger than those which do not.

§ 33. If Uranus could scarcely be detected from the motions of Neptune, much less can an extra-Neptunian planet, unless it happened to be nearly in conjunction with Neptune at the present time, and to have a much greater mass than Uranus, —a highly improbable combination of circumstances. That there is no present indication of any such action is shown by the smallness of the apparent mean errors of theory in heliocentric longitude and radius vector during the whole period from 1846 to 1864. The following table shows the mean value of these errors during each of the seven series of modern observations, and the error of the geocentric longitude of the Lalande observations, putting $\mu = 0$. The error of radius vector is expressed as error of annual parallax. It will be remembered that the first of the four equations of each series arise from observations made about half-way between the first quadrature and the opposition, the second at opposition, the third between opposition and last quadrature, and the fourth near the last quadrature. Each series, therefore, gives four equations of the first degree between the errors of heliocentric longitude δv, and annual parallax δp.

The coefficient of δv will be sensibly unity, and that of δp will vary from about -0.5 to $+1.0$ in each series.

Error of theory by the Lalande observations.

$+2''.3$

(It will be remembered that the probable error of the Lalande position was estimated at $2''.8$; but, owing to the over-estimate of the comparative precision of the modern observations, the weight assigned to this position in the equations of condition corresponded to a probable error of rather more than $4''$.)

By modern observations.

Limiting dates.	Error of longitude.	Error of parallax.
	$''$	$''$
1846–47,	-0.05	-0.18
1848–50,	-0.08	-0.03
1851–53,	-0.07	$+0.55$
1854–56,	-0.08	0.00
1857–59,	$+0.22$	$+0.23$
1860–62,	$+0.11$	$+0.18$
1863–64,	$+0.02$	$+0.28$

These errors are as small as could be expected if the theory were perfect. There is, therefore, no indication of the action of an extra-Neptunian planet. But this fact does not militate against the existence of such a planet. The perturbations of a planet, and its elliptic elements, develop themselves, not in proportion to the time, but in proportion to the square of the arc described. In order, therefore, to determine the errors of a slow-moving planet with as much accuracy as those of a quick-moving one, we must observe it through a period proportioned to its time of revolution. And we cannot detect a deviation of long period from an elliptic orbit until we have accumulated data much more than sufficient for the exact determination of the elliptic elements. For example, when the position of Neptune was determined from the perturbations of Uranus, the latter planet had been regularly observed through an arc of some 270°. Moreover, the two planets had been in conjunction in 1824. They are also remarkably near each other when in conjunction. Yet, with all these circumstances so favorable to the development of large perturbations, Uranus only wandered about $5''$ from an elliptic orbit during the entire period of the modern observations.

Perturbations will, at first, be developed in proportion to the square of the arc passed over. Therefore, had Uranus been observed through an arc of only 120°, the perturbations by Neptune would have been indicated only by deviations in heliocentric longitude of less than $1''$. It is, therefore, almost vain to hope for the detection of an extra-Neptunian planet from the motions of Neptune before the close of the present century.

§ 34. *Determination of the position of the plane of the orbit of Neptune.*

To determine the corrections of the constants p and q, which determine the

position of the plane of the orbit, we shall divide the residuals of latitude into five groups, the last one including three years, and each of the others four years. To find the heliocentric angular distance of the planet above the plane of its assumed orbit, we shall take an indiscriminate mean of the errors of geocentric latitude of each group, multiply it by 0.98 to reduce it to heliocentric error, and correct it for the mean error in longitude.

The mean errors of geocentric latitude, with the equations to which they give rise, are as follows. The probable errors of each modern mean is estimated at 0″.15: so that the Lalande position is entitled to a precision of $\frac{1}{10}$.

Limiting Dates.	$\delta\beta$	Equation of Condition.		
	″			″
1795,	+ 1.1	$0 = + 0.081\delta p$	$- 0.058\delta q$	+ 0.11
1846–49,	— 0.97	— 0.866	— 0.500	— 0.96
1850–53,	— 0.75	— 0.934	— 0.358	— 0.75
1854–57,	— 0.71	— 0.978	— 0.208	— 0.71
1858–61,	— 0.67	— 0.999	— 0.052	— 0.68
1862–64,	— 0.79	— 0.996	+ 0.084	— 0.80

The solution of which by least squares gives

$$\delta p = -0''.73; \quad \delta q = -0''.41.$$

The residuals, multiplying the first by 10 to reduce it to actual observed error, are

	″
1795,	+ 0.7
1846–49,	— 0.13
1850–53,	+ 0.07
1854–57,	+ 0.09
1858–61,	+ 0.07
1862–64,	— 0.10

So that the Lalande observation is represented within 0″.7, notwithstanding the small weight with which it enters the equations. In fact, if p and q were determined from the modern observations alone, the Lalande position would still be represented within about 0″.7.

§ 35. *Concluded elements of Neptune.*

From equations (1) and (2) of this chapter, we have

$$\delta\varepsilon = \delta x + 1.77\ \delta h \ + 0.85\ \delta k;$$
$$\delta n = \delta x' + 0.018\delta h \ - 0.073\delta k;$$

So that, making the mass of Uranus $\frac{1}{21000}$, the concluded corrections to the provisional elements of § 19 are

$$\begin{aligned}
\delta\varepsilon &= +\ 12''.94 \\
\delta n &= +\ \ 0.5144 \\
\delta h &= +\ \ 8.76 \\
\delta k &= -\ \ 3.79 \\
\delta p &= -\ \ 0.73 \\
\delta q &= -\ \ 0.41
\end{aligned}$$

Applying these corrections to the provisional elements of § 19, they become

$$\begin{aligned}
\varepsilon &= 335^\circ\ 5'\ 38''.91 \\
n &= 7864.9354 \\
h &= +1201.69 \\
k &= +1275.57 \\
p &= +4909.44 \\
q &= -4137.87
\end{aligned}$$

CHAPTER V.

TABLES OF NEPTUNE.

§ 36. *Fundamental theory.*

The fundamental theory on which these tables are founded is as follows:

1. *Undisturbed elements of Neptune, referred to the mean ecliptic and equinox of the epoch.*

$$
\begin{aligned}
h &= \text{eccentricity} \times \text{sine perihelion} = +\ 1201''.69 \\
k &= \text{eccentricity} \times \text{cos perihelion} = +\ 1275.57 \\
p &= \text{sine inclination} \times \text{sine node} = +\ 4909.44 \\
q &= \text{sine inclination} \times \text{cos node} = -\ 4137.87 \\
n &= \text{mean motion in } 365\tfrac{1}{4} \text{ days} = 7864.935 \\
\varepsilon &= \text{mean longitude at epoch} = 335^\circ\ 5'\ 38''.91
\end{aligned}
$$

Epoch 1850, Jan. 0, Greenwich mean noon.

From these expressions we deduce

$$
\begin{aligned}
\pi &= 43^\circ\ 17'\ 30''.3 \\
e &= 0.0084962 \\
\log a &= 1.4781414 \\
\text{Period} &= 164.782 \text{ Julian years.}
\end{aligned}
$$

In log a we have included the constants of log r introduced by the action of the planets, and also the effect of the secular variation of the longitude of the epoch, both of which are computed on p. 31.

2. *Secular and long-period perturbations of the above elements.*

These are taken without change from the table p. 39.

The elements being corrected by the addition of these perturbations for the epoch of computation, we thence deduce the elliptic place of the planet.

3. *Perturbations of the co-ordinates.*

To the elliptic place of the planet we apply corrections for periodic perturbations of the co-ordinates, as follows:

To the longitude in orbit,

$$P_{s.1} \sin l + P_{c.1} \cos l + P_{s.2} \sin 2\,l + P_{c.2} \cos 2\,l + \delta v_0.$$

To the logarithm of the radius vector,

$$R_{s.1} \sin l + R_{c.1} \cos l + \delta r_0.$$

To the north latitude, computed with the true longitude in orbit,

$$B_{s.1} \sin v + B_{c.1} \cos v + \delta\beta_0.$$

All these quantities have the same values as in § 19, pp. 40 and 41.

The elliptic values of the co-ordinates being thus corrected, we have the heliocentric co-ordinates resulting from the concluded theory.

To facilitate this computation, the following tables are constructed. They are designed to give the means of determining, for any date between the years 1600 and 2000, the principal auxiliary quantities which will be needed in computing the place of the planet from the above theory. Many of these quantities are modified so that the computer shall be troubled as little as possible with difference of signs. Thus, to all the quantities P_s, P_c, R_s, etc. constants are added so that they shall always be positive, and so that the signs of the products which form the perturbations shall be the same as those of sin l, cos l, etc. Again, constants are added to all the perturbations of the longitude and radius vector, to make them positive.

§ 37. *Data given in the several tables.*

Table I. gives the values of the "epochs and arguments" for the beginning of each fourth year from 1800 to 1952 inclusive, the years 1800 and 1900 beginning with Greenwich mean noon of Jan. 0, and all the other years with that of Jan. 1.

P is simply the number of the four-year cycle before 1900, by which l' and θ' of the next table must be multiplied, or $\frac{1900 - Y}{4}$, adding a unit for fractions.

l is the mean longitude in orbit of Neptune, affected with the long-period perturbations of that element, p. 39, and referred to the mean equinox of 1850.0.

y is the negative of the longitude of the node affected by perturbations, counted on the orbit of the planet from that point which is equally distant from the node of 1850 with the equinox of 1850, and diminished by 1°, the sum of the constants added to the equations of longitude.

θ is the longitude of the node, referred to the mean equinox of the epoch, and diminished by 1′, the constant added to the reduction to the ecliptic.

In the arguments 1 to 9 inclusive, the circle is divided into 400 parts. Representing the mean longitude of a planet, referred to the equinox of 1850.0 by its initial letter, the values of the different arguments are as follows:

$$
\begin{aligned}
\text{Arg. } 1 &= U - N, \\
\text{“ } \quad 2 &= S - N, \\
\text{“ } \quad 3 &= J - N, \\
\text{“ } \quad 4 &= 2S - N, \\
\text{“ } \quad 5 &= S, \\
\text{“ } \quad 6 &= S - 2N, \\
\text{“ } \quad 7 &= 2J - N, \\
\text{“ } \quad 8 &= J, \\
\text{“ } \quad 9 &= J - 2N.
\end{aligned}
$$

Thus, Arg. 1 gives the difference of the mean longitudes of Uranus and Neptune, expressed in parts 100 of which make a quadrant; and so of the other arguments.

At the bottom of the table the expression $\Delta_{(180)}^{(1)}$ is the change in the longitude or the argument during that 180 days which commences with 1850, Jan. 0.

Fact. T gives the change in $\Delta^{(1)}_{(180)}$ during a century: so that the change in any 180-day period within one or two centuries of the epoch may be found by multiplying Fact. T by the fraction of a century after 1850.0 at which the 180-day period commences, and applying it to $\Delta^{(1)}_{(180)}$.

$\Delta^{(2)}_{(180)}$ gives the second difference for any series of 180-day periods within one or two centuries of 1850: so that, knowing the first value of $\Delta^{(1)}_{(180)}$, we can find a series of values by successive addition.

The period of 180 days has been selected as a convenient one for computing a heliocentric ephemeris. If any other period, represented by N days, be preferred, the corresponding values of $\Delta^{(1)}$ and $\Delta^{(2)}$ are found by multiplying

$$\Delta^{(1)}_{(180)} \text{ by } \frac{N}{180},$$

and

$$\Delta^{(2)}_{(180)} \text{ by } \frac{N^2}{180^2}.$$

TABLE II. gives the change of each longitude and argument for the first day of each month during a four-year cycle. The change in l is given for that cycle which begins with 1900 and ends with 1904. Column l' gives, in units of the second decimal of seconds, the change in column l during one cycle. Hence, multiplying l' by the whole number P of the preceding table, and adding the units of the product to the hundredths of seconds of l, we have the change of mean longitude during the cycle numbered P in Table I. The correction is positive for years before 1900, because the mean motion is diminishing.

θ must be corrected in precisely the same way; but here the correction is negative before 1900.

Rigorously, both y and θ require correction similar to l. But it is not requisite that either of these quantities should be accurate within a second, so long as their sum is exactly equal to the precession diminished by 1° 1′. The four-year changes of both y and θ, which destroy each other, are, therefore, neglected; but the change in θ due to the secular variation of the constant of precession (0″.0227) is allowed for by the correction $P\theta'$.

TABLE III. gives the reduction from the first to the subsequent days of any month, or the motion of the epochs and arguments during a number of days one less than those on the left of the table.

TABLE IV. gives the corrections to be applied to the longitudes and arguments for the epochs $1800 + t$ to reduce them to the epochs $1600 + t$, $1700 + t$, and $1900 + t$, respectively. They are expressed in the form

$$a_0 + T \times \text{Fact. } T + T^2 \times \text{Fact. } T^2,$$

in which T is the fraction of a century.

TABLE V. gives the expressions for the perturbations of the longitude produced by Uranus. To each of the expressions $P_{s.1}$ and $P_{c.1}$ 14″ has been added, and to $P_{s.2}$ and $P_{c.2}$ 3″ has been added. Hence, when these quantities, as given in the

tables, are multiplied by sin l, cos l, sin 2 l, and cos 2 l, the sum will be too great by the quantity

$$14'' \sin l + 14'' \cos l + 3'' \sin 2\,l + 3'' \cos 2\,l,$$

which expression has been subtracted from the equation of the centre. The constant 14″ has been added to δv_1.

TABLE VI. gives the principal perturbations of the longitude produced by Saturn, namely,

$$\begin{aligned} & 18''.552 \sin\ (S-N) \\ - & \ 0\ .141 \sin 2\,(S-N) \\ - & \ 0\ .012 \sin 3\,(S-N) \\ + & \ (\text{const.} = 19''.000) \end{aligned}$$

TABLE VII. gives the principal perturbations of the longitude produced by Jupiter, namely,

$$\begin{aligned} & 34''.121 \sin\ (J-N) \\ - & \ 0\ .011 \sin 2\,(J-N) \\ + & \ (\text{const.} = 35''.000) \end{aligned}$$

TABLE VIII. gives the term

$$\begin{aligned} & -0''.524 \cos\,(2\,S-N) \\ & + (\text{const.} = 0''.600) \end{aligned}$$

TABLE IX. gives the terms

$$\begin{aligned} & -0''.058 \sin S \quad + 0''.047 \cos S \\ & + (\text{const.} = 0''.100) \end{aligned}$$

TABLE X. gives the terms

$$\begin{aligned} & + 0''.166 \sin\,(S-2\,N) \quad + 0''.436 \cos\,(S-2\,N) \\ & + (\text{const.} = 0''.500) \end{aligned}$$

TABLE XI. gives the terms

$$\begin{aligned} & + 0''.783 \sin\,(2\,J-N) \quad - 0''.164 \cos\,(2\,J-N) \\ & + (\text{const.} = 1''.100) \end{aligned}$$

TABLE XII. gives the terms

$$\begin{aligned} & -0''.101 \sin J \quad + 0''.097 \cos J \\ & + (\text{const.} = 0''.200) \end{aligned}$$

TABLE XIII. gives the terms

$$\begin{aligned} & + 0''.326 \sin\,(J-2\,N) \quad + 0''.297 \cos\,(J-2\,N) \\ & + (\text{const.} = 0''.500) \end{aligned}$$

TABLE XIV. will be more easily understood after we have explained the table of equation of the centre.

TABLE XV. is composed of the four following parts:

1. The equation of the centre in the undisturbed ellipse of 1850.0, or,

$$
\begin{array}{llll}
+ 2551''.117 \sin\ l & - 2403''.358 \cos\ l \\
+ \quad 1\ .163 \sin 2\,l & - \quad 18\ .580 \cos 2\,l \\
- \quad 0\ .088 \sin 3\,l & - \quad 0\ .104 \cos 3\,l
\end{array}
$$

2. The change in the equation of the centre produced by the perturbations of the elements h and k during that revolution of the planet which commenced 1779, Jan. 4, and ends 1943, Oct. 15. This change is represented by

$$2\,\delta k \sin l - 2\,\delta h \cos l,$$

δh and δk being taken from the table on p. 39 for the times corresponding to the various values of l during the period in question.

3. The terms

$$
\begin{array}{ll}
- 14'' \sin\ l & - 14'' \cos\ l \\
- \ 3 \ \sin 2\,l & - \ 3 \ \cos 2\,l
\end{array}
$$

introduced to destroy the effect of the constants added to the values of $P_{s.1}$, $P_{c.1}$, $P_{s.2}$, and $P_{c.2}$ to render them positive.

4. The constant

$$3529'',$$

added to render all the numbers of the table positive.

During the revolution to which Table XV. corresponds, the planet passed from 180° mean longitude, and returned to the same point in the heavens; whence the table begins and ends with this value of l. But since the commencement of the table corresponds to the values of h and k in 1779, and the end to these values in 1943, they do not correspond with each other. The sum of the constants added to Tables V. to XV. inclusive is 1°, which has been subtracted from y in Table I.

Table XIV. is formed by subtracting the values of δh and δk during the revolution of Table XV. from the values of the same elements 164.78 years earlier or later. Or, we have

$$
\begin{array}{l}
\Delta P_{s.1} = \quad 2\,(\delta k' - \delta k_0) \\
\Delta P_{c.1} = -\,2\,(\delta h' - \delta h_0)
\end{array}
$$

$\delta h'$ and $\delta k'$ representing the values of δh and δk at any epoch, and δh_0 and δk_0 their values at that date of the period 1779–1943 when the planet had the same mean longitude as at the epoch in question.

The sum of the sixteen quantities $P_{s.1} \sin l, P_{c.1} \cos l, P_{s.2} \sin 2\,l, P_{c.2} \cos 2\,l, \delta v\ ({}_1 \text{ to } {}_9)$, l, y, and the equation of Table XV. will give the true distance of the planet from its ascending node, which we represent by u.

Table XVI. gives the reduction to the ecliptic for the years 1800, 1900, and 2000, together with the change of the reduction for a century. The constant

$$60''$$

has been added to render all the numbers of the table positive.

The sum of u, θ, and the reduction to the ecliptic gives the true ecliptic longitude of the planet, referred to the mean equinox of the date.

Tables of the radius vector.

TABLE XVII. gives the values of

$$R_{s.1} + 150, \text{ and } R_{c.1} + 100.$$

The expressions for $R_{s.1}$ and $R_{c.1}$ are given on p. 40, § 19, and the units are those of the seventh place of decimals. $R_{s.1} + 150$ must be multiplied by $\sin l$, and $R_{c.1} + 100$ by $\cos l$, and the products included in the perturbations of $\log r$.

TABLE XVIII. gives the principal terms of the perturbations of the logarithm of the radius vector produced by Uranus, as given on p. 41. The constant added is 209.

TABLE XIX. gives the perturbations of the same element by Saturn, namely,

$$\begin{aligned} &397 \cos\ (S - N) \\ + &\ \ 4 \cos 2\ (S - N) \\ + &\ (\text{const.} = 400) \end{aligned}$$

TABLE XX. gives the perturbations of the same element by Jupiter, namely,

$$\begin{aligned} &701 \cos\ (J - N) \\ + &\ (\text{const.} = 700) \end{aligned}$$

The units of these tables are those of the seventh place of decimals.

TABLE XXI. is formed of the four following quantities.

1. A constant formed by applying the necessary corrections to the logarithm of the mean distance. We have

Mean motion, including its perturbations,	7864.935
Secular var. long. epoch,	+ 21.443
Elliptic mean motion,	7843.492
To which corresponds	$\log a = 1.4787334$
Constants of perturbations of $\log r$ (p. 31),	— 5920
Negative of constants added to Tables XVIII.–XX.,	— 1309
Constant to be substituted for $\log a$ in expression for log radius vector,	1.4780105

2. The elliptic $\log r - \log a$, namely,

$$\begin{aligned} &+ .0000078 \\ &- .0026857 \cos\ l \quad - .0025301 \sin\ l \\ &- .0000014 \cos 2\,l \quad - .0000235 \sin 2\,l \end{aligned}$$

3. The effects of the perturbations of h, k and a during the same revolution to which Table XV. corresponds, represented by

$$-\frac{M\delta h}{\sin 1''} \sin l \quad - \frac{M\delta k}{\sin 1''} \cos l + \delta \log a,$$

M being the modulus of the common system of logarithms.

4. The terms

$$-150 \sin l \quad -100 \cos l$$

introduced to destroy the effects of the constants added to $R_{s.1}$ and $R_{c.1}$.

TABLE XXII. gives the values of $B_{s.1}$ and $B_{c.1}$ (p. 40). The constant 0''.30 has been added to each of these quantities to render them positive.

TABLES XXIII. and XXIV. give the perturbations of the latitude produced by Saturn and Jupiter respectively, no constants being added.

TABLE XXV. gives the values of log sin i, to be added to log sin u in order to obtain the elliptic latitude. They, as well as θ, have been obtained from the formulæ

$$\sin i \sin \theta = p + \delta p + 0''.30$$
$$\sin i \cos \theta = q + \delta q - 0\ .30$$

The values of δp and δq being taken from the table p. 39, and the corrections $\pm$ 0''.30 being applied to destroy the effect of the constants added to $B_{s.1}$ and $B_{c.1}$.

§ 38. *Elementary precepts for the use of the tables.*

Express the date for which the position of Neptune is required, in years, months, and days of Greenwich mean time, according to the Gregorian Calendar.

If the date is between 1800 and 1955 inclusive, enter Table I. with the year, or the first preceding year found therein, and take out the values of l, y, θ, and Arguments 1–9 inclusive. Note also the value of P. If the date is not between the above limits, enter as if the number of the century were 18.

Enter Table II. with the excess of the actual year above that with which Table I. was entered, and with the month. Write the values of l, y, θ, and the arguments under those from Table I. Multiply l' and θ', the former interpolated to the day of the month, by P of Table I., and write the units of the product under the hundredths of seconds of l and θ, paying attention to the algebraic signs.

Enter Table III. with the day of the month, and write down l, &c., under the former values.

If the date is without the limits 1800–1955, enter Table IV. with the century, write the principal quantities under their proper heads, as before; multiply column "Fact. T" by the entire fraction of the century represented by the date, and column "Fact. T^2" by the square of this fraction, and write the products under their proper heads.

Add up all the partial values of l, y, θ, and the arguments thus obtained, attending to the algebraic signs of the products, subtracting from the arguments as many times 400 as possible, and we have the final values of those quantities.

Enter Table V. with the final value of Arg. 1, and take from it the five quantities there found. Multiply the first four of them as follows, using logarithms or natural numbers as may be most convenient:

$P_{s.1}$ by sine of l,
$P_{c.1}$ by cosine of l,
$P_{s.2}$ by sine of $2\,l$,
$P_{c.2}$ by cosine of $2\,l$.

But if the date is earlier than 1779 or later than 1943, $P_{s.1}$ and $P_{c.1}$ must first be corrected from Table XIV.

Write these four products under each other, remembering that their algebraic signs will be the same as those of the sine and cosine of l and $2\,l$, unless the corrections make $P_{s.1}$ or $P_{c.1}$ negative. Write under them the fifth quantity, δv_1.

Enter Tables VI. to XIII. inclusive, with the arguments at the top of each. Take out the eight remaining values of δv.

Enter Table XV. with l, first reducing the minutes and seconds to decimals of a degree, and take out the corresponding equation by interpolation to second differences.

Under these fourteen quantities write l and y, add up the sixteen lines, and call the sum u.

Under u write θ; enter Table XVI. with u (reduced to hundredths of a degree) as the side argument, and the year as the top argument, and take out the reduction to the ecliptic. Add it to u and θ, and the sum will be the heliocentric longitude of Neptune referred to the mean equinox and ecliptic of the date.

Enter Table XVII. with argument 1, and take out the values of $R_{s.1}$ and $R_{c.1}$. If the date is previous to 1779 or subsequent to 1943, multiply the values of $\Delta P_{s.1}$ and $\Delta P_{c.1}$ from Table XIV. by 10.53, and correct $R_{s.1}$ and $R_{c.1}$ as follows:

$$R_{s.1} \text{ by } \quad 10.53\,\Delta P_{c.1},$$
$$R_{c.1} \text{ by } -10.53\,\Delta P_{s.1},$$

adding the units of these products to the last figures of $R_{s.1}$ and $R_{c.1}$. Then multiply

$$R_{s.1} \text{ by sine of } l,$$
$$R_{c.1} \text{ by cosine of } l,$$

and write down the products with the algebraic sign of sine l and cos l respectively.

Enter Tables XVIII. to XX. with their proper arguments, and write the results under the products thus found.

Enter Table XXI. with the argument l, and take out the corresponding number, the first two figures of which are at the top of each column. Write it so that the last figure (the seventh place of decimals) shall be under the last figures of the former numbers.

The sum of the six numbers thus found will be the common logarithm of the radius vector of Neptune.

Enter Table XXII. with argument 1, and take out $B_{s.1}$ and $B_{c.1}$. Multiply the former by sin l and the latter by cos l.

Enter Tables XXIII. and XXIV. with their proper arguments, and take out the corresponding numbers, applying the proper algebraic signs.

Take the sine of i from Table XXV., and multiply it by the sine of u (u having already been found).

The sum of the five quantities thus found, each taken with its proper algebraic sign, will be the north latitude of Neptune above the plane of the ecliptic of the date.

Thus we shall have the heliocentric co-ordinates of the planet. The computer can then pass to the geocentric place by the method which he prefers.

If an ephemeris is wanted during a series of years, it will not be necessary to

take the arguments from Tables I.–IV. more than once in three or four, or even five, years. The intervals of computation are first to be chosen, and need not be less than 180 days for the heliocentric place. Then compute the values of l, y, θ, and the arguments for the first date of the series, and again for a date an integral number of intervals (not generally exceeding ten) later. The longitudes and arguments for the intermediate dates may then be found by continual addition of the differences for 180 days (if this is the interval) from the bottom of Table I.

§ 39. *Examples of the use of the tables.*

As a first example, we will compute an ephemeris of the heliocentric positions of Neptune for the years 1865 to 1868 inclusive. The intervals of computation will be 180 days, and we commence with the date 1864, Oct. 13, and end with 1869, March 21, between which are nine of the assumed intervals. We first compute the epochs and arguments for the extreme dates as follows:

1. For 1864, October 13.

	l	y	θ	Arg. 1	2
Table II., 1864,	5 40 58.30	228 54 54.37	130 15 49.25	91.96	199.49
Table III., Year 0, Oct.,	1 38 19.86	7.88	29.82	1.75	8.37
Fact. × 9,	.14		—.01		
Table IV., Day 13,	4 18.39	0.35	1.31	0.08	0.37
Epochs & Args. 1864, Oct. 13,	7 23 36.69	228 55 2.60	130 16 20.37	93.79	208.23

Arg.	3	4	5	6	7	8	9
Table II., 1864,	243.86	5	206	193	93.9	250	237
Table III., Year 0, Oct.,	23.47	19	10	7	48.8	25	22
Table IV., Day 13,	1.03	1	0	0	2.2	1	1
For 1864, Oct. 13.	268.36	25	216	200	144.9	276	260

2. For 1869, March 21.

	l	y	θ	Arg. 1	2
Table II., 1868,	14 25 17.73	228 55 36.39	130 18 28.26	101.30	244.11
Table III., Year 1, March,	2 32 31.25	12.23	46.25	2.72	12.98
Fact. × 8,	.19		—.01		
Table IV., Day 21,	7 10.64	0.57	2.18	0.13	0.61
For 1869, March 21,	17 4 59.81	228 55 49.19	130 19 16.68	104.15	257.70

Arg.	3	4	5	6	7	8	9
Table II., 1868,	369.03	104	260	228	353.9	385	353
Table III., Year 1, March,	36.41	29	16	10	75.7	39	34
Table IV., Day 21,	1.71	1	1	0	3.6	2	2
For 1869, March 21,	7.15	134	277	238	33.2	26	389

The epochs and arguments for the intermediate dates are now formed by successive additions of the change in 180 days, deduced from Table I. T, the fraction of a century after 1850, being 0.148, the first differences for 180 days, with the arguments, are found to be as follows:

	$\triangle_{180}$	1864, Oct. 13	1865, Apr. 11	1866, Apr. 6	1868, Sept. 22	1869, Mar. 21
	° ′ ″					
l	1 4 35.908	7 23 36.69	8 28 12.598	9 32 48.505	16 0 23.920	17 4 59.818
$\triangle l$	—.0012	1 4 35.908	1 4 35.907	1 4 35.905	1 4 35.898	
y	5.177	228 55 2.60	228 55 7.777	228 55 18.131	228 55 44.013	228 55 49.189
θ	19.590	130 16 20.37	130 16 39.960	130 16 59.550	130 18 57.093	130 19 16.684
Arg. 1	1.150	93.79	94.940	96.090	102.990	104.140
2	5.497	208.23	213.727	219.224	252.206	257.703
3	15.421	268.36	283.781	299.202	391.729	7.151
4	12.20	25.	37.2	49.4	122.6	134.8
5	6.7	216.	222.7	229.4	269.6	276.3
6	4.3	200.	204.3	208.6	234.4	238.7
7	32.03	144.9	176.93	208.96	1.14	33.17
8	16.6	276.	292.6	309.2	8.8	25.4
9	14.2	260.	274.2	288.4	373.6	387.8
		° ′	° ′	° ′	° ′	° ′
$2l$		14 47	16 56	19.6	32 1	34 10
		°	°	°	°	°
l (in Dec. of deg.)		7.3935	8.4702	9.5468	16.0066	17.0833

LONGITUDE.

	″	″	″	″	″
$P_{s.1}$	23.76	24.04	24.30	25.57	25.72
$P_{c.1}$	22.50	22.14	21.77	19.35	18.93
$P_{s.2}$	4.75	4.67	4.59	4.02	3.92
$P_{c.2}$	1.76	1.67	1.59	1.21	1.17
$P_{s.1} \sin l$	3.06	3.54	4.03	7.05	7.55
$P_{c.1} \cos l$	22.32	21.90	21.47	18.60	18.09
$P_{s.2} \sin 2l$	1.21	1.36	1.50	2.13	2.20
$P_{c.2} \cos 2l$	1.70	1.60	1.51	1.03	0.96
δv_1	11.15	11.49	11.83	13.79	14.12
δv_2	16.57	15.01	13.41	5.30	4.26
δv_3	5.03	1.98	0.88	30.58	38.83
δv_4	0.12	0.17	0.23	0.78	0.87
δv_5	0.07	0.07	0.08	0.13	0.14
δv_6	0.06	0.05	0.04	0.04	0.05
δv_7	1.80	1.53	1.15	0.95	1.35
δv_8	0.26	0.29	0.31	0.28	0.25
δv_9	0.06	0.08	0.13	0.64	0.73
Tab. XV.	0 23 59.17	0 24 53.01	0 25 47.59	0 31 29.74	0 32 29.03
l	7 23 36.69	8 28 12.60	9 32 48.50	16 0 23.92	17 4 59.81
y	228 55 2.60	228 55 7.78	228 55 12.96	228 55 44.01	228 55 49.19
u	236 43 41.87	237 49 12.46	238 54 45.62	245 28 58.97	246 34 47.43
θ	130 16 20.37	130 16 39.96	130 16 59.55	130 18 57.09	130 19 16.68
Red. Ecl.	14.23	15.01	15.88	22.33	23.62
Longitude	7 0 16.47	8 6 7.43	9 12 1.05	15 48 18.39	16 54 27.73

Radius Vector.

$R_{s.1}$	22	24	26	41	44
$R_{c.1}$	155	158	163	176	178
$R_{s.1} \sin l$	3		4	11	13
$R_{c.1} \cos l$	154	156	158	169	170
$\delta \log r_1$	82	77	73	48	44
$\delta \log r_2$	10	17	23	129	154
$\delta \log r_3$	366	524	691	1394	1396
Prin. term	1.4750064	1.4749650	1.4749250	1.4747074	1.4746754
$\log r$	1.4750679	1.4750427	1.4750199	1.4748825	1.4748531

Latitude.

$\log \sin u$	9.922246	9.927565	9.932667	9.958964	9.962660
$\log \sin i$	8.492852	8.492842	8.492831	8.492764	8.492753
$\log \sin \beta_0$	8.415098	8.420407	8.425498	8.451728	8.455413
	″	″	″	″	″
$B_{s.1}$	0.47	0.46	0.45	0.38	0.37
$B_{c.1}$	0.01	0.00	0.00	0.00	0.00
	″	″	″	″	″
$B_{s.1} \sin l$	+ 0.05	+ 0.06	+ 0.07	+ 0.10	+ 0.11
$B_{c.1} \cos l$	+ 0.01	0.00	0.00	0.00	0.00
$\delta\beta_1$	+ 0.28	+ 0.26	+ 0.24	+ 0.08	+ 0.05
$\delta\beta_2$	— 0.54	— 0.56	— 0.55	+ 0.12	+ 0.25
β_0	— 1 29 25.02	— 1 30 31.03	— 1 31 35.09	— 1 37 17.28	— 1 38 7.04
Latitude	— 1 29 25.22	— 1 30 31.27	— 1 31 35.33	— 1 37 16.98	— 1 38 6.63

Inserting the results for the five middle dates, the computations of which have been omitted in printing, for want of space, we have the following heliocentric ephemeris of Neptune:

Date.	Longitude (mean equinox of date).	Logarithm of radius vector.	Latitude.
	° ′ ″		° ′ ″
1864, Oct. 13,	7 0 16.47	1.4750679	— 1 29 25.22
1865, Apr. 11,	8 6 7.43	1.4750427	— 1 30 31.27
Oct. 8,	9 12 1.05	1.4750199	— 1 31 35.33
1866, Apr. 6,	10 17 57.51	1.4749986	— 1 32 37.36
Oct. 3,	11 23 56.84	1.4749778	— 1 33 37.41
1867, Apr. 1,	12 29 58.92	1.4749567	— 1 34 35.41
Sept. 28,	13 36 3.52	1.4749342	— 1 35 31.38
1868, Mar. 26,	14 42 10.14	1.4749097	— 1 36 25.26
Sept. 22,	15 48 18.39	1.4748825	— 1 37 16.98
1869, Mar. 21,	16 54 27.73	1.4748531	— 1 38 6.63

These co-ordinates being interpolated to every ten days, and corrected for nutation, the geocentric co-ordinates may then be computed and corrected for aberration in the usual way.

As another example, let us compute the heliocentric position of Neptune for Greenwich mean noon of 1795, May 9, the epoch of the normal place derived from Lalande's two observations.

	l	y	θ	Arg. 1	Arg. 2
	° ′ ″				
Table I., 1892,	66 51 12.69	228 59 48.26	130 34 22.60	157.30	111.85
Table II., 3^y May,	7 16 23.32	0 0 35.01	0 2 12.33	7.77	37.13
$2 \times l'$	.13		—.01		
Table III., Day 9,	0 2 52.26	0.23	0.87	0.05	0.24
Table IV., 1700,	141 31 19.97	359 42 21.53	358 53 55.63	166.69	85.00
Fact. $T \times .9536$,	+ 45.60	+ 6.58	— 8.75	— .05	— .36
Fact. $T^2 \times .91$,	+ 0.22	0	0	+ .01	— .02
1795, May 9,	215 42 34.19	228 42 51.61	129 30 22.67	331.77	233.84

$2\,l = 71.25$
$l = 215.7095$

	Arg. 3	4	5	6	7	8	9
Table I., 1892,	320.05	298	186	38	314.2	394	246
Table II., 3^y May,	104.18	82	45	29	216.6	112	96
Table III., Day 9,	0.68	1	0	0	1.4	1	1
Table IV., 1700,	70.66	327	242	328	298.6	228	313
Fact. $T \times .9536$,	+ 0.12	— 1	0	— 1	+ 0.2	0	0
1795, May 9,	95.69	307	73	394	31.0	335	256

Longitude.

	″
$P_{s.1}$	16.69
$P_{c.1}$	0.38
$P_{s.2}$	5.16
$P_{c.2}$	1.97
$P_{s.1}$ sin l	— 9.74
$P_{c.1}$ cos l	— 0.31
$P_{s.2}$ sin 2 l	+ 4.90
$P_{c.2}$ cos 2 l	+ 0.63
δv_1	24.08
δv_2	9.48
δv_3	69.05
δv_4	0.54
δv_5	0.07
δv_6	0.92
δv_7	1.32
δv_8	0.34
δv_9	0.06
Eq. Cent.	1 7 1.63
l	215 42 34.19
y	228 42 51.61
u	85 34 8.77
θ	129 30 22.67
Red. Ecliptic	52.26
Long. (Mean Eq.)	215 5 23.70
Nutation	— 15.90
Long. (True Eq.)	215 5 7.80

Radius vector.

$R_{s.1}$	242
$R_{c.1}$	77
$R_{s.1}$ sin l	— 141
$R_{c.1}$ cos l	— 63
δr_1	234
δr_2	60
δr_3	747
Prin. term	1.4816441
log r	1.4817278

Latitude.

	″
$B_{s.1}$	0.39
$B_{c.1}$	0.73
log sin u	9.998700
log sin i	8.494395
log sin β_0	8.493095
	″
$B_{s.1}$ sin l	— 0.22
$B_{c.1}$ cos l	— 0.59
$\delta \beta_1$	— 0.06
$\delta \beta_2$	— 0.46
β_0	+ 1 47 0.82
Latitude	+ 1 46 59.49

TABLE I.

Epochs and Arguments for the Beginning of each Fourth Year from 1800 to 1952.

Year.	P	l	y	θ	1
1800	25	225 51 36.90	228 43 40.52	129 33 27.21	342.62
1804	24	234 35 57.56	228 44 22.73	129 36 5.97	351.95
1808	23	243 20 18.15	228 45 04.92	129 38 44.75	361.28
1812	22	252 4 38.66	228 45 47.10	129 41 23.54	370.62
1816	21	260 48 59.10	228 46 29.27	129 44 2.35	379.95
1820	20	269 33 19.46	228 47 11.42	129 46 41.18	389.28
1824	19	278 17 39.74	228 47 53.56	129 49 20.02	398.62
1828	18	287 1 59.94	228 48 35.69	129 51 58.88	7.95
1832	17	295 46 20.07	228 49 17.81	129 54 37.75	17.29
1836	16	304 30 40.12	228 49 59.92	129 57 16.64	26.62
1840	15	313 15 00.09	228 50 42.02	129 59 55.54	35.95
1844	14	321 59 19.98	228 51 24.11	130 2 34.45	45.29
1848	13	330 43 39.80	228 52 6.18	130 5 13.38	54.62
1852	12	339 27 59.54	228 52 48.24	130 7 52.33	63.96
1856	11	348 12 19.20	228 53 30.30	130 10 31.29	73.29
1860	10	356 56 38.79	228 54 12.34	130 13 10.26	82.63
1864	9	5 40 58.30	228 54 54.37	130 15 49.25	91.96
1868	8	14 25 17.73	228 55 36.39	130 18 28.26	101.30
1872	7	23 9 37.08	228 56 18.40	130 21 7.28	110.63
1876	6	31 53 56.36	228 57 0.40	130 23 46.31	119.96
1880	5	40 38 15.56	228 57 42.39	130 26 25.35	129.30
1884	4	49 22 34.68	228 58 24.36	130 29 4.42	138.63
1888	3	58 6 53.72	228 59 6.32	130 31 43.50	147.97
1892	2	66 51 12.69	228 59 48.26	130 34 22.60	157.30
1896	1	75 35 31.58	229 0 30.20	130 37 1.71	166.64
1900	0	84 19 28.87	229 1 12.09	130 39 40.75	175.97
1904	−1	93 3 47.60	229 1 54.01	130 42 19.89	185.30
1908	−2	101 48 6.26	229 2 35.92	130 44 59.04	194.64
1912	−3	110 32 24.84	229 3 17.82	130 47 38.20	203.97
1916	−4	119 16 43.35	229 3 59.71	130 50 17.38	213.31
1920	−5	128 1 1.78	229 4 41.58	130 52 56.58	222.64
1924	−6	136 45 20.13	229 5 23.44	130 55 35.80	231.98
1928	−7	145 29 38.40	229 6 5.29	130 58 15.02	241.31
1932	−8	154 13 56.60	229 6 47.14	131 0 54.26	250.65
1936	−9	162 58 14.72	229 7 28.99	131 3 33.51	259.98
1940	−10	171 42 32.77	229 8 10.80	131 6 12.77	269.32
1944	−11	180 26 50.73	229 8 52.61	131 8 52.06	278.65
1948	−12	189 11 8.62	229 9 34.41	131 11 31.35	287.99
1952	−13	197 55 26.43	229 10 16.20	131 14 10.66	297.32
		° ′ ″	″	″	
$\Delta^{(1)}_{(180)}$		1 4 35.943	5.182	19.583	1.150
Fact. T		−0.237	−0.033	+0.044	0.0
$\Delta^{(2)}_{(180)}$		−0.0012	−0.0002	+0.0002	0

TABLE I.

EPOCHS AND ARGUMENTS FOR THE BEGINNING OF EACH FOURTH YEAR FROM 1800 TO 1952 (Continued).

Year.	2	3	4	5	6	7	8	9
1800	285.66	241.09	22	137	35	333.1	92	390
1804	330.27	366.26	121	191	70	193.2	227	105
1808	374.88	91.44	220	245	104	53.2	362	221
1812	19.49	216.61	319	300	139	313.3	97	336
1816	64.10	341.78	18	354	174	173.3	232	52
1820	108.71	66.96	117	8	209	33.4	367	167
1824	153.32	192.13	216	63	244	293.4	101	283
1828	197.94	317.31	315	117	279	153.5	236	398
1832	242.55	42.48	14	171	314	13.5	371	114
1836	287.17	167.65	113	226	349	273.5	106	229
1840	331.78	292.82	212	280	384	133.6	241	345
1844	376.40	18.00	310	334	19	393.6	376	60
1848	21.02	143.17	9	388	54	253.7	111	176
1852	65.64	268.34	108	43	88	113.7	245	291
1856	110.25	393.51	207	97	123	373.8	380	6
1860	154.87	118.68	306	151	158	233.8	115	122
1864	199.49	243.86	5	206	193	93.9	250	237
1868	244.16	369.03	104	260	228	353.9	385	353
1872	288.73	94.20	203	314	263	214.0	120	68
1876	333.35	219.37	302	369	298	74.0	255	184
1880	377.98	344.54	1	23	333	334.0	390	299
1884	22.60	69.71	100	77	368	194.1	125	15
1888	67.22	194.88	199	132	3	54.1	259	130
1892	111.85	320.05	298	186	38	314.2	394	246
1896	156.47	45.22	397	240	72	174.2	129	361
1900	201.06	170.29	96	295	107	34.3	264	77
1904	245.69	295.46	195	349	142	294.3	399	192
1908	290.32	20.63	294	3	177	154.4	134	308
1912	334.94	145.80	393	58	212	14.4	269	23
1916	379.57	270.97	92	112	247	274.5	4	138
1920	24.20	396.14	191	166	282	134.5	138	254
1924	68.82	121.30	290	221	317	394.6	273	369
1928	113.45	246.47	389	275	352	254.6	8	85
1932	158.08	371.64	88	330	387	114.7	143	200
1936	202.72	96.81	187	384	22	374.7	278	316
1940	247.35	221.97	286	38	57	234.8	13	31
1944	291.98	347.14	385	92	92	94.8	148	147
1948	336.61	72.31	84	147	126	354.9	283	262
1952	381.25	197.47	182	201	162	214.9	17	378
$\Delta^{(1)}_{(180)}$	5.497	15.421	12.20	6.7	4.3	32.03	16.6	14.2
Fact. T	$+.001$	0	0	0	0	0	0	0
$\Delta^{(2)}_{(180)}$	0	0	0	0	0	0	0	0

TABLE II.

Reduction of the Epochs and Arguments to the First Day of each Month in a Cycle of Four Years.

	l	l'	y	θ	θ'	1
Year 0,	° ′ ″		″	′ ″		
Jan. 1,	0 0 0.00	0.00	0.00	0 00.00	0.00	0.00
Feb. 1,	0 11 7.50	0.16	0.89	0 3.37	— 0.01	0.20
Mar. 1,	0 21 31.94	0.32	1.72	0 6.53	— 0.01	0.38
Apr. 1,	0 32 39.44	0.48	2.61	0 9.90	— 0.02	0.58
May 1,	0 43 25.41	0.64	3.48	0 13.17	— 0.03	0.77
June 1,	0 54 32.92	0.80	4.37	0 16.54	— 0.04	0.97
July 1,	1 5 18.89	0.96	5.23	0 19.81	— 0.05	1.16
Aug. 1,	1 16 26.39	1.12	6.12	0 23.18	— 0.05	1.36
Sept. 1,	1 27 33.89	1.29	7.01	0 26.55	— 0.06	1.56
Oct. 1,	1 38 19.86	1.45	7.88	0 29.82	— 0.07	1.75
Nov. 1,	1 49 27.37	1.61	8.77	0 33.19	— 0.08	1.95
Dec. 1,	2 0 13.34	1.77	9.64	0 36.46	— 0.08	2.14
Year 1,						
Jan. 1,	2 11 20.84	1.93	10.53	0 39.83	— 0.09	2.34
Feb. 1,	2 22 28.34	2.09	11.42	0 43.20	— 0.10	2.54
Mar. 1,	2 32 31.25	2.25	12.23	0 46.25	— 0.11	2.72
Apr. 1,	2 43 38.75	2.41	13.12	0 49.62	— 0.11	2.91
May 1,	2 54 24.72	2.57	13.98	0 52.89	— 0.12	3.10
June 1,	3 5 32.22	2.73	14.87	0 56.26	— 0.13	3.30
July 1,	3 16 18.19	2.89	15.74	0 59.53	— 0.14	3.49
Aug. 1,	3 27 25.70	3.05	16.63	1 2.90	— 0.14	3.69
Sept. 1,	3 38 33.20	3.21	17.52	1 6.28	— 0.15	3.89
Oct. 1,	3 49 19.17	3.37	18.39	1 9.54	— 0.16	4.08
Nov. 1,	4 0 26.67	3.53	19.28	1 12.92	— 0.17	4.28
Dec. 1,	4 11 12.64	3.69	20.15	1 16.18	— 0.17	4.47
Year 2,						
Jan. 1,	4 22 20.15	3.85	21.04	1 19.56	— 0.18	4.67
Feb. 1,	4 33 27.65	4.01	21.93	1 22.93	— 0.19	4.87
Mar. 1,	4 43 30.55	4.17	22.74	1 25.98	— 0.20	5.05
Apr. 1,	4 54 38.06	4.33	23.63	1 29.35	— 0.20	5.24
May 1,	5 5 24.03	4.49	24.49	1 32.62	— 0.21	5.44
June 1,	5 16 31.53	4.65	25.38	1 35.99	— 0.22	5.63
July 1,	5 27 17.50	4.81	26.25	1 39.26	— 0.23	5.83
Aug. 1,	5 38 25.00	4.97	27.14	1 42.63	— 0.23	6.02
Sept. 1,	5 49 32.50	5.13	28.03	1 46.00	— 0.24	6.22
Oct. 1,	6 0 18.47	5.29	28.90	1 49.26	— 0.25	6.41
Nov. 1,	6 11 25.97	5.45	29.79	1 52.64	— 0.26	6.61
Dec. 1,	6 22 11.94	5.61	30.66	1 55.90	— 0.26	6.80
Year 3,						
Jan. 1,	6 33 19.44	5.77	31.55	1 59.27	— 0.27	7.00
Feb. 1,	6 44 26.95	5.93	32.44	2 2.64	— 0.28	7.20
Mar. 1,	6 54 29.85	6.09	33.25	2 5.69	— 0.29	7.38
Apr. 1,	7 5 37.35	6.25	34.14	2 9.06	— 0.29	7.58
May 1,	7 16 23.32	6.41	35.00	2 12.33	— 0.30	7.77
June 1,	7 27 30.82	6.57	35.80	2 15.70	— 0.31	7.97
July 1,	7 38 16.79	6.73	36.77	2 18.97	— 0.32	8.16
Aug. 1,	7 49 24.29	6.89	37.66	2 22.34	— 0.32	8.36
Sept. 1,	8 0 31.80	7.05	38.55	2 25.72	— 0.33	8.55
Oct. 1,	8 11 17.76	7.21	39.40	2 28.98	— 0.34	8.75
Nov. 1,	8 22 25.27	7.37	40.30	2 32.36	— 0.34	8.94
Dec. 1,	8 33 11.24	7.53	41.17	2 35.62	— 0.35	9.14

Columns l' and θ' interpolated to the day of the month must be multiplied by the integer, P, of Table I. (not interpolated), and the units of the product added to the hundredths of seconds of l.

TABLE II.

Reduction of the Epochs and Arguments to the First Day of each Month in a Cycle of Four Years (Continued).

	2	3	4	5	6	7	8	9
Year 0,								
Jan. 1,	0.00	0.00	0	0	0	0.0	0	0
Feb. 1,	0.95	2.66	2	1	1	5.5	3	2
Mar. 1,	1.83	5.14	4	2	1	10.7	6	5
Apr. 1,	2.78	7.80	6	3	2	16.2	8	7
May 1,	3.70	10.37	8	4	3	21.6	11	10
June 1,	4.64	13.02	10	6	4	27.1	14	12
July 1,	5.56	15.59	12	7	4	32.4	17	14
Aug. 1,	6.50	18.25	14	8	5	37.9	20	17
Sept. 1,	7.45	20.90	16	9	6	43.5	22	19
Oct. 1,	8.37	23.47	19	10	7	48.8	25	22
Nov. 1,	9.31	26.13	21	11	7	54.3	28	24
Dec. 1,	10.23	28.70	23	12	8	59.7	31	26
Year 1,								
Jan. 1,	11.18	31.36	25	14	9	65.2	34	29
Feb. 1,	12.12	34.01	27	15	9	70.7	37	31
Mar. 1,	12.98	36.41	29	16	10	75.7	39	34
Apr. 1,	13.92	39.07	31	17	11	81.3	42	36
May 1,	14.84	41.64	33	18	12	86.6	45	38
June 1,	15.79	44.29	35	19	12	92.2	48	41
July 1,	16.70	46.86	37	20	13	97.5	50	43
Aug. 1,	17.65	49.52	39	22	14	103.0	53	46
Sept. 1,	18.51	52.18	41	23	15	108.5	56	48
Oct. 1,	19.51	54.75	43	24	15	113.9	59	51
Nov. 1,	20.46	57.40	45	25	16	119.4	62	53
Dec. 1,	21.38	59.97	47	26	17	124.7	65	55
Year 2,								
Jan. 1,	22.32	62.63	50	27	17	130.2	68	58
Feb. 1,	23.27	65.29	52	28	18	135.7	70	60
Mar. 1,	24.12	67.68	54	29	19	140.7	73	62
Apr. 1,	25.07	70.34	56	30	20	146.2	76	65
May 1,	25.99	72.91	58	32	20	151.6	79	67
June 1,	26.93	75.57	60	33	21	157.1	81	70
July 1,	27.85	78.14	62	34	22	162.4	84	72
Aug. 1,	28.80	80.79	64	35	23	167.9	87	74
Sept. 1,	29.74	83.45	66	36	23	173.4	90	77
Oct. 1,	30.66	86.02	68	37	24	178.8	93	79
Nov. 1,	31.60	88.67	70	38	25	184.3	96	82
Dec. 1,	32.52	91.24	72	40	25	189.6	98	84
Year 3,								
Jan. 1,	33.47	93.90	74	41	26	195.1	101	87
Feb. 1,	34.42	96.56	76	42	27	200.7	104	89
Mar. 1,	35.27	98.96	78	43	28	205.7	107	91
Apr. 1,	36.22	101.61	80	44	28	211.2	110	94
May 1,	37.13	104.18	82	45	29	216.6	112	96
June 1,	38.08	106.84	84	46	30	222.1	115	98
July 1,	39.00	109.41	87	47	30	227.4	118	01
Aug. 1,	39.94	112.06	89	49	31	232.9	121	03
Sept. 1,	40.89	114.72	91	50	32	238.4	124	106
Oct. 1,	41.81	117.29	93	51	33	243.8	126	108
Nov. 1,	42.75	119.95	95	52	34	249.3	129	111
Dec. 1,	43.67	122.52	97	53	34	254.6	132	113

TABLE III.

Reduction from the First to Subsequent Days of any Month.

Days.	l	y	θ	1	2	3	4	5	6	7	8	9
	′ ″	″	″									
1	0 0.00	0.00	0.00	0.00	0.00	0.00	0	0	0	0.0	0	0
2	0 21.53	0.03	0.11	0.01	0.03	0.09	0	0	0	0.2	0	0
3	0 43.06	0.06	0.22	0.01	0.06	0.17	0	0	0	0.4	0	0
4	1 4.60	0.09	0.33	0.02	0.09	0.26	0	0	0	0.5	0	0
5	1 26.13	0.11	0.44	0.03	0.12	0.34	0	0	0	0.7	0	0
6	1 47.66	0.14	0.54	0.03	0.15	0.43	0	0	0	0.9	0	0
7	2 9.19	0.17	0.65	0.04	0.18	0.51	0	0	0	1.1	0	0
8	2 30.73	0.20	0.76	0.04	0.21	0.60	1	0	0	1.3	1	1
9	2 52.26	0.23	0.87	0.05	0.24	0.68	1	0	0	1.4	1	1
10	3 13.79	0.26	0.98	0.06	0.27	0.77	1	0	0	1.6	1	1
11	3 35.32	0.29	1.09	0.06	0.30	0.86	1	0	0	1.8	1	1
12	3 56.86	0.32	1.20	0.07	0.34	0.94	1	0	0	2.0	1	1
13	4 18.39	0.35	1.31	0.08	0.37	1.03	1	0	0	2.2	1	1
14	4 39.92	0.37	1.41	0.08	0.40	1.11	1	0	0	2.3	1	1
15	5 1.45	0.40	1.52	0.09	0.43	1.20	1	1	0	2.5	1	1
16	5 22.99	0.43	1.63	0.10	0.46	1.28	1	1	0	2.7	1	1
17	5 44.52	0.46	1.74	0.10	0.49	1.37	1	1	0	2.9	2	1
18	6 6.05	0.49	1.85	0.11	0.52	1.46	1	1	0	3.1	2	1
19	6 27.58	0.52	1.96	0.12	0.55	1.54	1	1	0	3.2	2	1
20	6 49.11	0.54	2.07	0.12	0.58	1.63	1	1	0	3.4	2	1
21	7 10.65	0.57	2.18	0.13	0.61	1.71	1	1	0	3.6	2	2
22	7 32.18	0.60	2.29	0.13	0.64	1.80	1	1	1	3.8	2	2
23	7 53.71	0.63	2.39	0.14	0.67	1.88	2	1	1	3.9	2	2
24	8 15.24	0.66	2.50	0.15	0.70	1.97	2	1	1	4.1	2	2
25	8 36.78	0.69	2.61	0.15	0.73	2.06	2	1	1	4.3	2	2
26	8 58.31	0.72	2.72	0.16	0.76	2.14	2	1	1	4.4	2	2
27	9 19.84	0.75	2.83	0.17	0.79	2.23	2	1	1	4.6	2	2
28	9 41.37	0.78	2.94	0.17	0.83	2.31	2	1	1	4.8	3	2
29	10 2.91	0.80	3.05	0.18	0.86	2.40	2	1	1	4.9	3	2
30	10 24.44	0.83	3.16	0.18	0.89	2.48	2	1	1	5.1	3	2
31	10 45.97	0.86	3.26	0.19	0.92	2.57	2	1	1	5.3	3	2

In January and February of 1700, 1800, and 1900, Table III. must be entered with a number of days 1 *greater* than the real day of the month.

TABLE IV.

Corrections for Past and Future Centuries.

	1600	Fact. T	Fact. T^2	1700	Fact. T	Fact. T^2	1900	Fact. T	Fact. T^2
	° ′ ″	″	″	° ′ ″	″	″	° ′ ″	″	″
l	283 1 52.88	+94.09	+1.03	141 31 19.97	+47.82	+0.24	218 27 51.97	−48.17	+0.17
y	359 24 35.91	+14.05	0	359 42 21.53	+ 6.90	0	0 17 31.57	− 6.92	0
θ	357 48 0.68	−18.59	0	358 53 55.63	− 9.17	0	1 6 13.54	+ 9.19	0
Arg. 1	333.41	−0.08	+0.01	166.69	−0.05	+0.01	233.35	+0.02	−0.10
2	170.36	−0.69	−0.08	85.00	−0.38	−0.02	315.40	+0.46	+0.06
3	141.19	+0.22	+0.04	70.66	+0.13	0.00	329.20	−0.17	0
4	255.	−2	0	327.	−1	0	74.	0	0
5	85.	−1	0	242.	0	0	158.	0	0
6	256.	−1.	0	328.	−1	0	72.	+1.	0
7	196.9	+0.5	0	298.6	+0.2	0	101.2	−0.4	0
8	55.	0	0	228.	0	0	172.	0	0
9	227.	0	0	313.	0	0	87.		0

Table.	V.								VI.		VII.	
Arg.	1								2		3	
	$P_{s.1}$	Diff.	$P_{c.1}$	Diff.	$P_{s.2}$	$P_{c.2}$	δv_1	Diff.	δv_2	Diff.	δv_3	Diff.
	″	″	″	″	″	″	″	″	″	″	″	″
0	0.38		13.22		0.54	2.84	13.85		19.00		35.00	
1	0.36	0.02	13.49	0.27	0.54	2.92	13.33	0.52	19.29	0.29	35.54	0.54
2	0.35	0.01	13.76	0.27	0.53	3.00	12.80	0.53	19.57	0.28	36.07	0.53
3	0.34	0.01	14.03	0.27	0.53	3.08	12.28	0.52	19.86	0.29	36.61	0.54
4	0.33	0.01	14.30	0.27	0.53	3.16	11.76	0.52	20.14	0.28	37.14	0.53
5	0.33	0.00	14.57	0.27	0.54	3.25	11.25	0.51	20.43	0.29	37.68	0.54
6	0.33	0.00	14.84	0.27	0.54	3.33	10.74	0.51	20.72	0.29	38.21	0.53
7	0.34	0.01	15.11	0.27	0.55	3.41	10.24	0.50	21.00	0.28	38.75	0.54
8	0.34	0.00	15.38	0.27	0.56	3.50	9.74	0.50	21.29	0.29	39.28	0.53
9	0.36	0.02	15.66	0.28	0.58	3.58	9.25	0.49	21.57	0.28	39.81	0.53
10	0.38	0.02	15.93	0.27	0.59	3.66	8.76	0.49	21.85	0.28	40.34	0.53
11	0.40	0.02	16.21	0.28	0.61	3.74	8.29	0.47	22.14	0.29	40.87	0.53
12	0.43	0.03	16.49	0.28	0.64	3.83	7.83	0.46	22.42	0.28	41.39	0.52
13	0.46	0.03	16.76	0.27	0.66	3.91	7.38	0.45	22.70	0.28	41.92	0.53
14	0.50	0.04	17.04	0.28	0.69	3.99	6.94	0.44	22.98	0.28	42.44	0.52
15	0.55	0.05	17.32	0.28	0.72	4.07	6.51	0.43	23.26	0.28	42.97	0.53
16	0.60	0.05	17.60	0.28	0.76	4.15	6.09	0.42	23.54	0.28	43.49	0.52
17	0.66	0.06	17.88	0.28	0.80	4.23	5.69	0.40	23.81	0.27	44.01	0.52
18	0.73	0.07	18.17	0.29	0.84	4.31	5.30	0.39	24.09	0.28	44.53	0.52
19	0.81	0.08	18.45	0.28	0.88	4.39	4.93	0.37	24.37	0.28	45.04	0.51
20	0.89	0.08	18.74	0.29	0.93	4.46	4.57	0.36	24.64	0.27	45.55	0.51
21	0.98	0.09	19.02	0.28	0.97	4.54	4.22	0.35	24.91	0.27	46.06	0.51
22	1.08	0.10	19.31	0.29	1.03	4.61	3.89	0.33	25.18	0.27	46.57	0.51
23	1.19	0.11	19.59	0.28	1.08	4.68	3.58	0.31	25.45	0.27	47.07	0.50
24	1.30	0.11	19.88	0.29	1.14	4.76	3.28	0.30	25.72	0.27	47.57	0.50
25	1.43	0.13	20.16	0.28	1.20	4.83	3.00	0.28	25.99	0.27	48.07	0.50
26	1.56	0.13	20.44	0.28	1.27	4.89	2.73	0.27	26.25	0.26	48.56	0.49
27	1.70	0.14	20.73	0.29	1.34	4.96	2.48	0.25	26.52	0.27	49.05	0.49
28	1.86	0.16	21.01	0.28	1.41	5.02	2.25	0.23	26.78	0.26	49.54	0.49
29	2.02	0.16	21.29	0.28	1.49	5.08	2.03	0.22	27.04	0.26	50.02	0.48
30	2.19	0.17	21.57	0.28	1.56	5.14	1.82	0.21	27.30	0.26	50.50	0.48
31	2.38	0.19	21.84	0.27	1.64	5.20	1.63	0.19	27.55	0.25	50.97	0.47
32	2.57	0.19	22.11	0.27	1.73	5.25	1.46	0.17	27.81	0.26	51.44	0.47
33	2.77	0.20	22.38	0.27	1.81	5.30	1.30	0.16	28.06	0.25	51.91	0.47
34	2.98	0.21	22.65	0.27	1.90	5.34	1.16	0.14	28.31	0.25	52.38	0.47
35	3.21	0.23	22.91	0.26	1.99	5.38	1.04	0.12	28.56	0.25	52.84	0.46
36	3.44	0.23	23.17	0.26	2.09	5.42	0.93	0.11	28.80	0.24	53.29	0.45
37	3.68	0.24	23.42	0.25	2.18	5.45	0.84	0.09	29.04	0.24	53.74	0.45
38	3.93	0.25	23.67	0.25	2.28	5.48	0.76	0.08	29.28	0.24	54.18	0.44
39	4.19	0.26	23.92	0.25	2.38	5.51	0.69	0.07	29.52	0.24	54.63	0.45
40	4.46	0.27	24.16	0.24	2.48	5.53	0.64	0.05	29.76	0.24	55.07	0.44
41	4.74	0.28	24.39	0.23	2.58	5.54	0.60	0.04	29.99	0.23	55.50	0.43
42	5.03	0.29	24.62	0.23	2.69	5.56	0.58	0.02	30.22	0.23	55.92	0.42
43	5.33	0.30	24.84	0.22	2.79	5.56	0.57	0.01	30.45	0.23	56.34	0.42
44	5.64	0.31	25.05	0.21	2.90	5.56	0.57	0.00	30.68	0.23	56.76	0.42
45	5.95	0.31	25.25	0.20	3.00	5.56	0.59	0.02	30.90	0.22	57.17	0.41
46	6.28	0.33	25.45	0.20	3.11	5.55	0.61	0.02	31.12	0.22	57.57	0.40
47	6.61	0.33	25.64	0.19	3.21	5.54	0.65	0.04	31.34	0.22	57.97	0.40
48	6.94	0.33	25.82	0.18	3.32	5.53	0.70	0.05	31.55	0.21	58.36	0.39
49	7.29	0.35	25.99	0.17	3.42	5.51	0.77	0.07	31.76	0.21	58.75	0.39
50	7.64	0.35	26.16	0.17	3.52	5.48	0.85	0.08	31.97	0.21	59.14	0.39

Table.	V.								VI.		VII.	
Arg.	1								2		3	
	$P_{s.1}$	Diff.	$P_{c.1}$	Diff.	$P_{s.2}$	$P_{c.2}$	δv_1	Diff.	δv_2	Diff.	δr_3	Diff.
	"	"	"	"	"	"	"	"	"	"	"	"
50	7.64		26.16		3.52	5.48	0.85		31.97		59.14	
51	8.00	0.36	26.31	0.15	3.63	5.45	0.94	0.09	32.17	0.20	59.52	0.38
52	8.36	0.36	26.45	0.14	3.73	5.41	1.03	0.09	32.38	0.21	59.89	0.37
53	8.73	0.37	26.59	0.14	3.83	5.37	1.14	0.11	32.58	0.20	60.25	0.36
54	9.11	0.38	26.71	0.12	3.92	5.33	1.26	0.12	32.77	0.19	60.61	0.36
55	9.49	0.38	26.82	0.11	4.02	5.28	1.39	0.13	32.96	0.19	60.96	0.35
56	9.87	0.38	26.93	0.11	4.11	5.23	1.53	0.14	33.15	0.19	61.30	0.34
57	10.26	0.39	27.02	0.09	4.20	5.17	1.67	0.14	33.34	0.19	61.63	0.33
58	10.65	0.39	27.10	0.08	4.29	5.11	1.83	0.16	33.52	0.18	61.96	0.33
59	11.05	0.40	27.17	0.07	4.38	5.04	2.00	0.17	33.70	0.18	62.29	0.33
60	11.45	0.40	27.23	0.06	4.46	4.97	2.18	0.18	33.87	0.17	62.61	0.32
61	11.85	0.40	27.28	0.05	4.55	4.90	2.37	0.19	34.04	0.17	62.91	0.30
62	12.25	0.40	27.32	0.04	4.62	4.82	2.56	0.19	34.21	0.17	63.22	0.31
63	12.65	0.40	27.34	0.02	4.70	4.74	2.76	0.20	34.37	0.16	63.52	0.30
64	13.06	0.41	27.36	0.02	4.76	4.65	2.97	0.21	34.53	0.16	63.81	0.29
65	13.47	0.41	27.36	0.00	4.83	4.57	3.18	0.21	34.69	0.16	64.10	0.29
66	13.88	0.41	27.35	0.01	4.89	4.48	3.40	0.22	34.84	0.15	64.38	0.28
67	14.29	0.41	27.33	0.02	4.94	4.39	3.63	0.23	34.99	0.15	64.65	0.27
68	14.70	0.41	27.29	0.04	5.00	4.29	3.87	0.24	35.14	0.15	64.91	0.26
69	15.10	0.40	27.25	0.04	5.04	4.20	4.11	0.24	35.28	0.14	65.17	0.26
70	15.51	0.41	27.19	0.06	5.09	4.10	4.36	0.25	35.42	0.14	65.41	0.24
71	15.92	0.41	27.12	0.07	5.13	3.99	4.61	0.25	35.55	0.13	65.65	0.24
72	16.32	0.40	27.04	0.08	5.16	3.89	4.86	0.25	35.68	0.13	65.88	0.23
73	16.72	0.40	26.95	0.09	5.19	3.79	5.13	0.27	35.81	0.13	66.10	0.22
74	17.12	0.40	26.84	0.11	5.22	3.69	5.39	0.26	35.93	0.12	66.32	0.22
75	17.51	0.39	26.72	0.12	5.24	3.58	5.66	0.27	36.04	0.11	66.53	0.21
76	17.90	0.39	26.59	0.13	5.25	3.48	5.94	0.28	36.16	0.12	66.72	0.19
77	18.29	0.39	26.45	0.14	5.26	3.37	6.21	0.27	36.27	0.11	66.92	0.20
78	18.67	0.38	26.30	0.15	5.27	3.27	6.50	0.29	36.37	0.10	67.11	0.19
79	19.05	0.38	26.14	0.16	5.27	3.16	6.78	0.28	36.47	0.10	67.28	0.17
80	19.42	0.37	25.96	0.18	5.27	3.05	7.07	0.29	36.57	0.10	67.45	0.17
81	19.78	0.36	25.78	0.18	5.26	2.95	7.36	0.29	36.66	0.09	67.61	0.16
82	20.14	0.36	25.58	0.20	5.25	2.85	7.65	0.29	36.75	0.09	67.76	0.15
83	20.49	0.35	25.37	0.21	5.23	2.74	7.94	0.29	36.83	0.08	67.91	0.15
84	20.84	0.35	25.15	0.22	5.21	2.64	8.24	0.30	36.91	0.08	68.05	0.14
85	21.18	0.34	24.93	0.22	5.18	2.54	8.54	0.30	36.98	0.07	68.18	0.13
86	21.51	0.33	24.69	0.24	5.15	2.45	8.83	0.29	37.05	0.07	68.30	0.12
87	21.83	0.32	24.44	0.25	5.11	2.35	9.13	0.30	37.12	0.07	68.41	0.11
88	22.14	0.31	24.18	0.26	5.07	2.25	9.43	0.30	37.18	0.06	68.52	0.11
89	22.45	0.31	23.91	0.27	5.02	2.16	9.73	0.30	37.24	0.06	68.61	0.09
90	22.74	0.29	23.64	0.27	4.97	2.07	10.02	0.29	37.29	0.05	68.70	0.09
91	23.02	0.28	23.35	0.29	4.92	1.98	10.32	0.30	37.34	0.05	68.78	0.08
92	23.30	0.28	23.06	0.29	4.86	1.90	10.62	0.30	37.38	0.04	68.85	0.07
93	23.56	0.26	22.75	0.31	4.80	1.82	10.92	0.30	37.42	0.04	68.91	0.06
94	23.81	0.25	22.44	0.31	4.74	1.74	11.21	0.29	37.45	0.03	68.97	0.06
95	24.05	0.24	22.12	0.32	4.67	1.66	11.51	0.30	37.48	0.03	69.02	0.05
96	24.28	0.23	21.80	0.32	4.60	1.59	11.80	0.29	37.51	0.03	69.06	0.04
97	24.50	0.22	21.47	0.33	4.53	1.53	12.10	0.30	37.53	0.02	69.09	0.03
98	24.71	0.21	21.13	0.34	4.45	1.46	12.38	0.28	37.55	0.02	69.11	0.02
99	24.91	0.20	20.78	0.35	4.37	1.40	12.67	0.29	37.56	0.01	69.12	0.01
100	25.10	0.19	20.43	0.35	4.28	1.35	12.96	0.29	37.56	0.00	69.12	0.00

Table.	V.								VI.		VII.	
Arg.	1								2		3	
	$P_{s.1}$	Diff.	$P_{c.1}$	Diff.	$P_{s.1}$	$P_{c.2}$	δv_1	Diff.	δv_2	Diff.	δv_3	Diff.
	"	"	"	"	"	"	"	"	"	"	"	"
100	25.10		20.43		4.28	1.35	12.96		37.56		69.12	
101	25.26	0.16	20.08	0.35	4.20	1.30	13.24	0.28	37.57	0.01	69.12	0.00
102	25.42	0.16	19.72	0.36	4.11	1.25	13.52	0.28	37.56	0.01	69.11	0.01
103	25.57	0.15	19.35	0.37	4.02	1.21	13.79	0.27	37.56	0.00	69.09	0.02
104	25.70	0.13	18.98	0.37	3.93	1.17	14.08	0.29	37.54	0.02	69.06	0.03
105	25.82	0.12	18.61	0.37	3.84	1.14	14.34	0.26	37.53	0.01	69.02	0.04
106	25.92	0.10	18.23	0.38	3.75	1.11	14.60	0.26	37.51	0.02	68.97	0.05
107	26.02	0.10	17.86	0.37	3.65	1.09	14.87	0.27	37.48	0.03	68.91	0.06
108	26.10	0.08	17.47	0.39	3.56	1.07	15.13	0.26	37.45	0.03	68.85	0.06
109	26.16	0.06	17.09	0.38	3.46	1.05	15.38	0.25	37.42	0.03	68.78	0.07
110	26.22	0.06	16.71	0.38	3.36	1.04	15.63	0.25	37.38	0.04	68.70	0.08
111	26.26	0.04	16.32	0.39	3.27	1.04	15.87	0.24	37.33	0.05	68.62	0.08
112	26.28	0.02	15.94	0.28	3.17	1.04	16.11	0.24	37.28	0.05	68.52	0.10
113	26.30	0.02	15.56	0.38	3.07	1.04	16.34	0.23	37.23	0.05	68.41	0.11
114	26.29	0.01	15.17	0.39	2.98	1.06	16.57	0.23	37.17	0.06	68.30	0.11
115	26.28	0.01	14.79	0.38	2.88	1.07	16.80	0.23	37.11	0.06	68.18	0.12
116	26.25	0.03	14.41	0.38	2.78	1.09	17.02	0.22	37.05	0.06	68.05	0.13
117	26.20	0.05	14.03	0.38	2.69	1.11	17.24	0.22	36.98	0.07	67.91	0.14
118	26.15	0.05	13.65	0.38	2.60	1.14	17.44	0.20	36.90	0.08	67.76	0.15
119	26.08	0.07	13.27	0.38	2.51	1.18	17.65	0.21	36.82	0.08	67.61	0.15
120	26.00	0.08	12.90	0.37	2.42	1.21	17.84	0.19	36.74	0.08	67.45	0.16
121	25.90	0.10	12.53	0.37	2.34	1.26	18.03	0.19	36.65	0.09	67.28	0.17
122	25.79	0.11	12.17	0.36	2.26	1.30	18.22	0.19	36.55	0.10	67.11	0.17
123	25.67	0.12	11.81	0.36	2.18	1.35	18.40	0.18	36.46	0.09	66.92	0.19
124	25.54	0.13	11.45	0.36	2.10	1.41	18.57	0.17	36.35	0.11	66.73	0.19
125	25.40	0.14	11.10	0.35	2.03	1.46	18.74	0.17	36.24	0.11	66.53	0.20
126	25.24	0.16	10.76	0.34	1.95	1.52	18.89	0.15	36.13	0.11	66.32	0.21
127	25.07	0.17	10.42	0.34	1.89	1.59	19.05	0.16	36.02	0.11	66.10	0.22
128	24.89	0.18	10.09	0.33	1.82	1.66	19.19	0.14	35.90	0.12	65.88	0.22
129	24.70	0.19	9.77	0.32	1.76	1.73	19.33	0.14	35.77	0.13	65.65	0.23
130	24.49	0.21	9.45	0.32	1.70	1.80	19.46	0.13	35.65	0.12	65.41	0.24
131	24.28	0.21	9.14	0.31	1.65	1.88	19.59	0.13	35.51	0.14	65.17	0.24
132	24.05	0.23	8.84	0.30	1.60	1.96	19.70	0.11	35.38	0.13	64.91	0.26
133	23.82	0.23	8.55	0.29	1.55	2.04	19.82	0.12	35.24	0.14	64.65	0.26
134	23.57	0.25	8.27	0.28	1.51	2.12	19.92	0.10	35.09	0.15	64.38	0.27
135	23.31	0.26	8.00	0.27	1.47	2.21	20.02	0.10	34.94	0.15	64.10	0.28
136	23.04	0.27	7.73	0.27	1.44	2.29	20.11	0.09	34.79	0.15	63.81	0.29
137	22.77	0.27	7.48	0.25	1.41	2.38	20.19	0.08	34.63	0.16	63.52	0.29
138	22.49	0.28	7.23	0.25	1.39	2.47	20.27	0.08	34.47	0.16	63.22	0.30
139	22.19	0.30	7.00	0.23	1.37	2.56	20.34	0.07	34.31	0.16	62.91	0.31
140	21.89	0.30	6.77	0.23	1.36	2.65	20.40	0.06	34.14	0.17	62.61	0.30
141	21.58	0.31	6.56	0.21	1.35	2.74	20.46	0.06	33.97	0.17	62.29	0.32
142	21.27	0.31	6.36	0.20	1.34	2.83	20.50	0.04	33.79	0.18	61.96	0.33
143	20.95	0.32	6.17	0.19	1.34	2.92	20.54	0.04	33.61	0.18	61.63	0.33
144	20.63	0.32	5.99	0.18	1.35	3.01	20.58	0.04	33.43	0.18	61.30	0.33
145	20.29	0.34	5.82	0.17	1.35	3.10	20.60	0.02	33.24	0.19	60.96	0.34
146	19.96	0.33	5.67	0.15	1.37	3.18	20.63	0.03	33.05	0.19	60.61	0.35
147	19.61	0.35	5.53	0.14	1.39	3.27	20.64	0.01	32.85	0.20	60.25	0.36
148	19.26	0.35	5.40	0.13	1.41	3.36	20.65	0.01	32.66	0.19	59.89	0.36
149	18.91	0.35	5.28	0.12	1.44	3.44	20.65	0.00	32.45	0.21	59.52	0.37
150	18.56	0.35	5.17	0.11	1.47	3.52	20.65	0.00	32.25	0.20	59.14	0.38

Table.	V.								VI.		VII.	
Arg.	1								2		3	
	$P_{s.1}$	Diff.	$P_{c.1}$	Diff.	$P_{s.2}$	$P_{c.2}$	δv_1	Diff.	δv_2	Diff.	δv_3	Diff.
	″	″	″	″	″	″	″	″	″	″	″	″
150	18.56		5.17		1.47	3.52	20.62		32.25		59.14	
151	18.20	0.36	5.08	0.09	1.50	3.60	20.64	0.01	32.04	0.21	58.75	0.39
152	17.84	0.36	5.00	0.08	1.54	3.68	20.62	0.02	31.83	0.21	58.36	0.39
153	17.47	0.37	4.93	0.07	1.59	3.76	20.59	0.03	31.62	0.21	57.97	0.39
154	17.10	0.37	4.87	0.06	1.63	3.83	20.56	0.03	31.40	0.22	57.57	0.40
155	16.74	0.36	4.83	0.04	1.69	3.90	20.52	0.04	31.18	0.22	57.17	0.40
156	16.37	0.37	4.80	0.03	1.74	3.97	20.48	0.04	30.95	0.23	56.76	0.41
157	16.00	0.37	4.79	0.01	1.80	4.03	20.43	0.05	30.73	0.22	56.34	0.42
158	15.64	0.36	4.78	0.01	1.85	4.09	20.37	0.06	30.50	0.23	55.92	0.42
159	15.27	0.37	4.79	0.01	1.92	4.15	20.31	0.06	30.26	0.24	55.50	0.42
160	14.90	0.37	4.82	0.03	1.98	4.21	20.24	0.07	30.03	0.23	55.06	0.44
161	14.54	0.36	4.85	0.03	2.05	4.26	20.16	0.08	29.79	0.24	54.63	0.43
162	14.18	0.36	4.90	0.05	2.12	4.30	20.08	0.08	29.55	0.24	54.18	0.45
163	13.82	0.36	4.97	0.07	2.20	4.35	19.99	0.09	29.31	0.24	53.74	0.44
164	13.46	0.36	5.04	0.07	2.27	4.39	19.90	0.09	29.06	0.25	53.29	0.45
165	13.11	0.35	5.13	0.09	2.35	4.42	19.80	0.10	28.81	0.25	52.84	0.45
166	12.76	0.35	5.23	0.10	2.43	4.45	19.70	0.10	28.56	0.25	52.38	0.46
167	12.42	0.34	5.34	0.11	2.51	4.48	19.59	0.11	28.30	0.26	51.91	0.47
168	12.08	0.34	5.46	0.12	2.59	4.50	19.48	0.11	28.04	0.26	51.44	0.47
169	11.74	0.34	5.60	0.14	2.67	4.52	19.36	0.12	27.79	0.25	50.97	0.47
170	11.41	0.33	5.74	0.14	2.75	4.53	19.24	0.12	27.52	0.27	50.50	0.47
171	11.09	0.32	5.90	0.16	2.83	4.54	19.11	0.13	27.26	0.26	50.02	0.48
172	10.77	0.32	6.07	0.17	2.92	4.54	18.98	0.13	27.00	0.26	49.54	0.48
173	10.46	0.31	6.25	0.18	3.00	4.54	18.84	0.14	26.73	0.27	49.05	0.49
174	10.16	0.30	6.44	0.19	3.08	4.53	18.70	0.14	26.46	0.27	48.56	0.49
175	9.87	0.29	6.65	0.21	3.16	4.53	18.55	0.15	26.19	0.27	48.07	0.49
176	9.58	0.29	6.86	0.21	3.25	4.51	18.40	0.15	25.92	0.27	47.57	0.50
177	9.31	0.27	7.09	0.23	3.33	4.49	18.24	0.16	25.64	0.28	47.07	0.50
178	9.04	0.27	7.32	0.23	3.41	4.47	18.09	0.15	25.37	0.27	46.57	0.50
179	8.78	0.26	7.56	0.24	3.49	4.44	17.92	0.17	25.09	0.28	46.06	0.51
180	8.53	0.25	7.82	0.26	3.57	4.41	17.76	0.16	24.81	0.28	45.54	0.52
181	8.29	0.24	8.08	0.26	3.64	4.38	17.59	0.17	24.52	0.29	45.03	0.51
182	8.06	0.23	8.35	0.27	3.71	4.34	17.41	0.18	24.24	0.28	44.52	0.51
183	7.84	0.22	8.62	0.27	3.79	4.30	17.24	0.17	23.96	0.28	44.01	0.51
184	7.63	0.21	8.91	0.29	3.85	4.25	17.06	0.18	23.67	0.29	43.49	0.52
185	7.43	0.20	9.20	0.29	3.92	4.20	16.88	0.18	23.39	0.28	42.97	0.52
186	7.24	0.19	9.49	0.29	3.98	4.14	16.70	0.18	23.10	0.29	42.44	0.53
187	7.06	0.18	9.80	0.31	4.04	4.09	16.51	0.19	22.81	0.29	41.92	0.52
188	6.90	0.16	10.11	0.31	4.10	4.03	16.32	0.19	22.52	0.29	41.39	0.53
189	6.75	0.15	10.42	0.31	4.15	3.96	16.13	0.19	22.23	0.29	40.87	0.52
190	6.61	0.14	10.75	0.33	4.20	3.90	15.94	0.19	21.94	0.29	40.34	0.53
191	6.48	0.13	11.07	0.32	4.25	3.83	15.75	0.19	21.65	0.29	39.81	0.53
192	6.36	0.12	11.40	0.33	4.29	3.76	15.55	0.20	21.36	0.29	39.28	0.53
193	6.26	0.10	11.74	0.34	4.33	3.69	15.36	0.19	21.06	0.30	38.75	0.53
194	6.17	0.09	12.08	0.34	4.36	3.61	15.16	0.20	20.77	0.29	38.21	0.54
195	6.10	0.07	12.42	0.34	4.39	3.54	14.96	0.20	20.47	0.30	37.68	0.53
196	6.03	0.07	12.76	0.34	4.42	3.46	14.76	0.20	20.18	0.29	37.14	0.54
197	5.98	0.05	13.11	0.35	4.44	3.38	14.56	0.20	19.89	0.29	36.61	0.53
198	5.94	0.04	13.46	0.35	4.46	3.30	14.36	0.20	19.59	0.30	36.07	0.54
199	5.92	0.02	13.80	0.34	4.48	3.22	14.16	0.20	19.30	0.29	35.54	0.53
200	5.91	0.01	14.16	0.36	4.49	3.13	13.96	0.20	19.00	0.30	35.00	0.54

TABLE.	V.								VI.		VII.	
Arg.	1								2		3	
	$P_{s.1}$	Diff.	$P_{c.1}$	Diff.	$P_{s.2}$	$P_{c.2}$	δv_1	Diff.	δv_2	Diff.	δv_3	Diff.
	″	″	″	″	″	″	″	″	″	″	″	″
200	5.91		14.16		4.49	3.13	13.96		19.00		35.00	
201	5.91	0.00	14.51	0.35	4.49	3.05	13.76	0.20	18.70	0.30	34.46	0.54
202	5.92	0.01	14.86	0.35	4.49	2.97	13.56	0.20	18.41	0.29	33.93	0.53
203	5.95	0.03	15.21	0.35	4.49	2.88	13.36	0.20	18.11	0.30	33.39	0.54
204	6.00	0.05	15.55	0.34	4.48	2.80	13.16	0.20	17.82	0.29	32.86	0.53
		0.06		0.35				0.19		0.29		0.54
205	6.06		15.90		4.47	2.72	12.97		17.53		32.32	
206	6.12	0.06	16.25	0.35	4.45	2.64	12.77	0.20	17.23	0.30	31.79	0.53
207	6.21	0.09	16.59	0.34	4.43	2.56	12.57	0.20	16.94	0.29	31.25	0.54
208	6.30	0.09	16.93	0.34	4.41	2.48	12.38	0.19	16.64	0.30	30.72	0.53
209	6.41	0.11	17.27	0.34	4.38	2.41	12.19	0.19	16.35	0.29	30.19	0.53
		0.12		0.33				0.19		0.29		0.53
210	6.53		17.60		4.34	2.33	12.00		16.06		29.66	
211	6.65	0.12	17.93	0.33	4.31	2.26	11.81	0.19	15.77	0.29	29.13	0.53
212	6.80	0.15	18.26	0.33	4.27	2.19	11.62	0.19	15.48	0.29	28.61	0.52
213	6.96	0.16	18.57	0.31	4.22	2.13	11.43	0.19	15.19	0.29	28.08	0.53
214	7.14	0.18	18.89	0.32	4.17	2.06	11.25	0.18	14.90	0.29	27.56	0.52
		0.18		0.31				0.18		0.29		0.53
215	7.32		19.20		4.12	2.00	11.07		14.61		27.03	
216	7.51	0.19	19.50	0.30	4.07	1.94	10.89	0.18	14.33	0.28	26.51	0.52
217	7.72	0.21	19.79	0.29	4.01	1.88	10.72	0.17	14.04	0.29	25.99	0.52
218	7.94	0.22	20.08	0.29	3.95	1.83	10.55	0.17	13.76	0.28	25.48	0.51
219	8.16	0.22	20.36	0.28	3.88	1.78	10.38	0.17	13.48	0.28	24.96	0.52
		0.24		0.27				0.16		0.29		0.51
220	8.40		20.63		3.82	1.73	10.22		13.19		24.45	
221	8.65	0.25	20.90	0.27	3.75	1.69	10.06	0.16	12.91	0.28	23.94	0.51
222	8.91	0.26	21.15	0.25	3.68	1.65	9.90	0.16	12.63	0.28	23.43	0.51
223	9.18	0.27	21.40	0.25	3.60	1.62	9.74	0.16	12.36	0.27	22.93	0.50
224	9.45	0.27	21.64	0.24	3.53	1.59	9.59	0.15	12.08	0.28	22.43	0.50
		0.29		0.23				0.15		0.27		0.50
225	9.74		21.87		3.45	1.56	9.44		11.81		21.93	
226	10.04	0.30	22.08	0.21	3.37	1.54	9.30	0.14	11.54	0.27	21.44	0.49
227	10.34	0.30	22.29	0.21	3.29	1.52	9.16	0.14	11.27	0.27	20.95	0.49
228	10.65	0.31	22.49	0.20	3.22	1.50	9.03	0.13	11.00	0.27	20.46	0.49
229	10.97	0.32	22.67	0.18	3.14	1.49	8.90	0.13	10.74	0.26	19.98	0.48
		0.33		0.18				0.12		0.26		0.48
230	11.30		22.85		3.06	1.49	8.78		10.48		19.50	
231	11.63	0.33	23.01	0.16	2.97	1.49	8.66	0.12	10.21	0.27	19.03	0.47
232	11.97	0.34	23.16	0.15	2.89	1.49	8.54	0.12	9.96	0.25	18.56	0.47
233	12.32	0.35	23.30	0.14	2.81	1.50	8.44	0.10	9.70	0.26	18.09	0.47
234	12.67	0.35	23.42	0.12	2.73	1.51	8.33	0.11	9.44	0.26	17.62	0.47
		0.36		0.12				0.09		0.25		0.46
235	13.03		23.54		2.65	1.53	8.24		9.19		17.16	
236	13.39	0.36	23.64	0.10	2.57	1.55	8.14	0.10	8.94	0.25	16.71	0.45
237	13.76	0.37	23.73	0.09	2.49	1.58	8.06	0.08	8.69	0.25	16.26	0.45
238	14.12	0.36	23.81	0.08	2.42	1.60	7.97	0.09	8.45	0.24	15.82	0.44
239	14.50	0.38	23.87	0.06	2.34	1.64	7.90	0.07	8.21	0.24	15.37	0.45
		0.37		0.06				0.07		0.24		0.43
240	14.87		23.93		2.27	1.68	7.83		7.97		14.94	
241	15.25	0.38	23.96	0.03	2.20	1.72	7.77	0.06	7.74	0.23	14.50	0.44
242	15.63	0.38	23.99	0.03	2.13	1.76	7.71	0.06	7.50	0.24	14.08	0.42
243	16.01	0.38	24.00	0.01	2.06	1.81	7.66	0.05	7.27	0.23	13.66	0.42
244	16.39	0.38	24.00	0.00	2.00	1.87	7.61	0.05	7.05	0.22	13.24	0.42
		0.38		0.01				0.04		0.23		0.41
245	16.77		23.99		1.93	1.92	7.57		6.82		12.83	
246	17.16	0.39	23.96	0.03	1.88	1.98	7.54	0.03	6.60	0.22	12.43	0.40
247	17.54	0.38	23.92	0.04	1.82	2.04	7.52	0.02	6.38	0.22	12.03	0.40
248	17.92	0.38	23.86	0.06	1.77	2.11	7.50	0.02	6.17	0.21	11.64	0.39
249	18.30	0.38	23.79	0.07	1.72	2.18	7.48	0.02	5.96	0.21	11.25	0.39
		0.37		0.08				0.00		0.21		0.39
250	18.67		23.71		1.67	2.25	7.48		5.75		10.86	

Table.	V.								VI.		VII.	
Arg.	1								2		3	
	$P_{s.1}$	Diff.	$P_{c.1}$	Diff.	$P_{s.2}$	$P_{c.2}$	δv_1	Diff.	δv_2	Diff.	δv_3	Diff.
	″	″	″	″	″	″	″	″	″	″	″	″
250	18.67		23.71		1.67	2.25	7.48		5.75		10.86	
251	19.05	0.38	23.62	0.09	1.63	2.32	7.47	0.01	5.55	0.20	10.48	0.38
252	19.42	0.37	23.51	0.11	1.59	2.40	7.48	0.01	5.34	0.21	10.11	0.37
253	19.78	0.36	23.39	0.12	1.55	2.48	7.50	0.02	5.15	0.19	9.75	0.36
254	20.15	0.37	23.26	0.13	1.52	2.56	7.52	0.02	4.95	0.20	9.39	0.36
255	20.51	0.36	23.11	0.15	1.50	2.64	7.54	0.02	4.76	0.19	9.04	0.35
256	20.86	0.35	22.96	0.15	1.48	2.72	7.58	0.04	4.57	0.19	8.70	0.34
257	21.21	0.35	22.78	0.18	1.46	2.81	7.61	0.03	4.39	0.18	8.37	0.33
258	21.55	0.34	22.60	0.18	1.44	2.90	7.66	0.05	4.21	0.18	8.04	0.33
259	21.89	0.34	22.41	0.19	1.43	2.98	7.71	0.05	4.03	0.18	7.71	0.33
260	22.22	0.33	22.21	0.20	1.43	3.07	7.77	0.06	3.86	0.17	7.39	0.32
261	22.55	0.33	21.99	0.22	1.43	3.16	7.84	0.07	3.69	0.17	7.09	0.30
262	22.86	0.31	21.76	0.23	1.43	3.25	7.91	0.07	3.53	0.16	6.78	0.31
263	23.17	0.31	21.52	0.24	1.44	3.33	7.99	0.08	3.37	0.16	6.48	0.30
264	23.47	0.30	21.27	0.25	1.46	3.42	8.08	0.09	3.21	0.16	6.19	0.29
265	23.76	0.29	21.01	0.26	1.47	3.51	8.17	0.09	3.06	0.15	5.90	0.29
266	24.04	0.28	20.74	0.27	1.50	3.59	8.28	0.11	2.91	0.15	5.62	0.28
267	24.32	0.28	20.46	0.28	1.52	3.68	8.38	0.10	2.76	0.15	5.35	0.27
268	24.58	0.26	20.17	0.29	1.56	3.76	8.50	0.12	2.62	0.14	5.09	0.26
269	24.83	0.25	19.87	0.30	1.59	3.85	8.62	0.12	2.49	0.13	4.83	0.26
270	25.08	0.25	19.57	0.30	1.63	3.93	8.74	0.12	2.35	0.14	4.59	0.24
271	25.31	0.23	19.25	0.32	1.68	4.01	8.88	0.14	2.23	0.12	4.35	0.24
272	25.53	0.22	18.93	0.32	1.72	4.08	9.02	0.14	2.10	0.13	4.12	0.23
273	25.74	0.21	18.59	0.34	1.78	4.16	9.17	0.15	1.98	0.12	3.90	0.22
274	25.93	0.19	18.26	0.33	1.83	4.23	9.32	0.15	1.87	0.11	3.68	0.22
275	26.12	0.19	17.91	0.35	1.89	4.30	9.48	0.16	1.76	0.11	3.47	0.21
276	26.29	0.17	17.56	0.35	1.96	4.36	9.65	0.17	1.65	0.11	3.27	0.20
277	26.46	0.17	17.20	0.36	2.02	4.43	9.82	0.17	1.54	0.11	3.08	0.19
278	26.60	0.14	16.84	0.36	2.09	4.49	10.00	0.18	1.45	0.09	2.89	0.19
279	26.74	0.14	16.47	0.37	2.16	4.55	10.19	0.19	1.35	0.10	2.72	0.17
280	26.87	0.13	16.09	0.38	2.24	4.60	10.38	0.19	1.26	0.09	2.55	0.17
281	26.98	0.11	15.71	0.38	2.32	4.65	10.57	0.19	1.18	0.08	2.39	0.16
282	27.07	0.09	15.33	0.38	2.40	4.69	10.78	0.21	1.10	0.08	2.24	0.15
283	27.16	0.09	14.94	0.39	2.48	4.73	10.98	0.20	1.02	0.08	2.09	0.15
284	27.23	0.07	14.55	0.39	2.57	4.77	11.20	0.22	0.95	0.07	1.95	0.14
285	27.29	0.06	14.16	0.39	2.66	4.80	11.42	0.22	0.89	0.06	1.82	0.13
286	27.34	0.05	13.77	0.39	2.75	4.83	11.64	0.22	0.83	0.06	1.70	0.12
287	27.37	0.03	13.37	0.40	2.84	4.85	11.87	0.23	0.77	0.06	1.59	0.11
288	27.38	0.01	12.97	0.40	2.93	4.87	12.10	0.23	0.71	0.06	1.48	0.11
289	27.39	0.01	12.58	0.39	3.03	4.88	12.34	0.24	0.67	0.04	1.38	0.10
290	27.38	0.01	12.18	0.40	3.12	4.89	12.59	0.25	0.62	0.05	1.30	0.08
291	27.35	0.03	11.78	0.40	3.21	4.90	12.84	0.25	0.58	0.04	1.22	0.08
292	27.31	0.04	11.38	0.40	3.31	4.90	13.09	0.25	0.55	0.03	1.15	0.07
293	27.26	0.05	10.98	0.40	3.40	4.90	13.34	0.25	0.52	0.03	1.09	0.06
294	27.20	0.06	10.58	0.40	3.50	4.89	13.60	0.26	0.49	0.03	1.03	0.06
295	27.12	0.08	10.19	0.39	3.59	4.87	13.87	0.27	0.47	0.02	0.98	0.05
296	27.03	0.09	9.80	0.39	3.69	4.85	14.13	0.26	0.46	0.01	0.94	0.04
297	26.92	0.11	9.41	0.39	3.78	4.83	14.40	0.27	0.44	0.02	0.91	0.03
298	26.80	0.12	9.02	0.39	3.87	4.80	14.68	0.28	0.44	0.00	0.89	0.02
299	26.67	0.13	8.64	0.38	3.96	4.76	14.95	0.27	0.43	0.01	0.88	0.01
300	26.53	0.14	8.26	0.38	4.06	4.73	15.23	0.28	0.44	0.01	0.88	0.00

Table.	V.								VI.		VII.	
Arg.	1								2		3	
	$P_{s.1}$	Diff.	$P_{c.1}$	Diff.	$P_{s.2}$	$P_{c.2}$	δv_1	Diff.	δv_2	Diff.	δv_3	Diff.
	″	″	″	″	″	″	″	″	″	″	″	″
300	26.53		8.26		4.06	4.73	15.23		0.44		0.88	
301	26.37	0.16	7.89	0.37	4.14	4.68	15.51	0.28	0.44	0.00	0.88	0.00
302	26.20	0.17	7.52	0.37	4.23	4.64	15.80	0.29	0.45	0.01	0.89	0.01
303	26.02	0.18	7.16	0.36	4.31	4.59	16.08	0.28	0.47	0.02	0.91	0.02
304	25.82	0.20	6.80	0.36	4.40	4.43	16.37	0.29	0.49	0.02	0.94	0.03
		0.20		0.35				0.28		0.03		0.04
305	25.62		6.45		4.47	4.47	16.65		0.52		0.98	
306	25.40	0.22	6.10	0.35	4.55	4.41	16.95	0.30	0.55	0.03	1.03	0.05
307	25.17	0.23	5.76	0.34	4.62	4.34	17.24	0.29	0.58	0.03	1.09	0.06
308	24.92	0.25	5.43	0.33	4.69	4.27	17.53	0.29	0.62	0.04	1.15	0.06
309	24.67	0.25	5.11	0.32	4.76	4.20	17.82	0.29	0.66	0.04	1.22	0.07
		0.26		0.31				0.30		0.05		0.08
310	24.41		4.80		4.83	4.12	18.12		0.71		1.30	
311	24.14	0.27	4.49	0.31	4.89	4.04	18.41	0.29	0.76	0.05	1.38	0.08
312	23.85	0.29	4.19	0.30	4.94	3.96	18.71	0.30	0.82	0.06	1.48	0.10
313	23.56	0.29	3.90	0.29	5.00	3.87	19.00	0.29	0.88	0.06	1.59	0.11
314	23.26	0.30	3.62	0.28	5.05	3.78	19.29	0.29	0.95	0.07	1.70	0.11
		0.32		0.27				0.29		0.07		0.12
315	22.94		3.35		5.09	3.69	19.58		1.02		1.82	
316	22.62	0.32	3.08	0.27	5.13	3.59	19.87	0.29	1.09	0.07	1.95	0.13
317	22.29	0.33	2.83	0.25	5.17	3.50	20.16	0.29	1.17	0.08	2.09	0.14
318	21.95	0.34	2.59	0.24	5.20	3.40	20.45	0.29	1.25	0.08	2.24	0.15
319	21.61	0.34	2.36	0.23	5.23	3.30	20.73	0.28	1.34	0.09	2.39	0.15
		0.36		0.22				0.29		0.09		0.16
320	21.25		2.14		5.25	3.20	21.02		1.43		2.55	
321	20.89	0.36	1.93	0.21	5.27	3.10	21.29	0.27	1.53	0.10	2.72	0.17
322	20.53	0.36	1.73	0.20	5.28	2.99	21.57	0.28	1.63	0.10	2.90	0.18
323	20.15	0.38	1.54	0.19	5.29	2.89	21.84	0.27	1.73	0.10	3.09	0.19
324	19.78	0.37	1.36	0.18	5.30	2.78	22.12	0.28	1.84	0.11	3.28	0.19
		0.39		0.16				0.26		0.12		0.20
325	19.39		1.20		5.29	2.68	22.38		1.96		3.48	
326	19.00	0.39	1.04	0.16	5.29	2.57	22.65	0.27	2.07	0.11	3.70	0.22
327	18.61	0.39	0.90	0.14	5.28	2.46	22.91	0.26	2.19	0.12	3.92	0.22
328	18.22	0.39	0.77	0.13	5.26	2.36	23.16	0.25	2.32	0.13	4.14	0.22
329	17.82	0.40	0.65	0.12	5.24	2.25	23.41	0.25	2.45	0.13	4.37	0.23
		0.41		0.11				0.25		0.13		0.24
330	17.41		0.54		5.22	2.15	23.66		2.58		4.61	
331	17.01	0.40	0.45	0.09	5.19	2.05	23.90	0.24	2.72	0.14	4.85	0.24
332	16.60	0.41	0.36	0.09	5.15	1.95	24.13	0.23	2.86	0.14	5.11	0.26
333	16.19	0.41	0.29	0.07	5.12	1.85	24.36	0.23	3.01	0.15	5.37	0.26
334	15.78	0.41	0.24	0.05	5.07	1.75	24.58	0.22	3.16	0.15	5.64	0.27
		0.41		0.05				0.22		0.15		0.28
335	15.37		0.19		5.02	1.65	24.80		3.31		5.92	
336	14.95	0.42	0.16	0.03	4.97	1.56	25.00	0.20	3.47	0.16	6.21	0.29
337	14.54	0.41	0.13	0.03	4.92	1.47	25.21	0.21	3.63	0.16	6.50	0.29
338	14.13	0.41	0.12	0.01	4.85	1.38	25.40	0.19	3.79	0.16	6.80	0.30
339	13.72	0.41	0.12	0.00	4.79	1.29	25.59	0.19	3.96	0.17	7.11	0.31
		0.41		0.01				0.17		0.17		0.30
340	13.31		0.13		4.72	1.21	25.76		4.13		7.41	
341	12.90	0.41	0.16	0.03	4.65	1.13	25.93	0.17	4.30	0.17	7.73	0.32
342	12.49	0.41	0.19	0.03	4.57	1.05	26.09	0.16	4.48	0.18	8.06	0.33
343	12.09	0.40	0.24	0.05	4.49	0.98	26.24	0.15	4.66	0.18	8.39	0.33
344	11.69	0.40	0.30	0.06	4.41	0.91	26.39	0.15	4.85	0.19	8.72	0.33
		0.40		0.07				0.13		0.19		0.34
345	11.29		0.37		4.32	0.85	26.52		5.04		9.06	
346	10.89	0.40	0.46	0.09	4.24	0.79	26.65	0.13	5.23	0.19	9.41	0.35
347	10.50	0.39	0.55	0.09	4.15	0.73	26.76	0.11	5.42	0.19	9.77	0.36
348	10.12	0.38	0.65	0.10	4.05	0.68	26.86	0.10	5.62	0.20	10.13	0.36
349	9.73	0.39	0.77	0.12	3.96	0.63	26.95	0.09	5.83	0.21	10.50	0.37
		0.38		0.12				0.08		0.20		0.38
350	9.35		0.89		3.86	0.59	27.03		6.03		10.88	

Table.	V.								VI.		VII.	
Arg.	1								2		3	
	$P_{s.1}$	Diff.	$P_{c.1}$	Diff.	$P_{s.2}$	$P_{c.2}$	δv_1	Diff.	δv_2	Diff.	δv_3	Diff.
	″	″	″	″	″	″	″	″	″	″	″	″
350	9.35		0.89		3.86	0.59	27.03		6.03		10.88	
351	8.98	0.37	1.02	0.13	3.77	0.55	27.09	0.06	6.24	0.21	11.26	0.38
352	8.62	0.36	1.17	0.15	3.66	0.52	27.15	0.06	6.45	0.21	11.66	0.40
353	8.26	0.36	1.32	0.15	3.56	0.49	27.20	0.05	6.66	0.21	12.05	0.39
354	7.90	0.36	1.48	0.16	3.46	0.47	27.23	0.03	6.88	0.22	12.45	0.40
355	7.56	0.34	1.66	0.18	3.36	0.45	27.26	0.03	7.10	0.22	12.85	0.40
356	7.22	0.34	1.83	0.17	3.26	0.43	27.27	0.01	7.32	0.22	13.26	0.41
357	6.89	0.33	2.02	0.19	3.15	0.42	27.27	0.00	7.55	0.23	13.68	0.42
358	6.56	0.33	2.22	0.20	3.05	0.42	27.25	0.02	7.78	0.23	14.10	0.42
359	6.25	0.31	2.42	0.20	2.95	0.42	27.22	0.03	8.01	0.23	14.52	0.42
360	5.94	0.31	2.63	0.21	2.85	0.42	27.18	0.04	8.24	0.23	14.95	0.43
361	5.64	0.30	2.85	0.22	2.75	0.43	27.12	0.06	8.48	0.24	15.39	0.44
362	5.35	0.29	3.07	0.22	2.64	0.44	27.05	0.07	8.72	0.24	15.84	0.45
363	5.06	0.29	3.30	0.23	2.54	0.46	26.97	0.08	8.96	0.24	16.28	0.44
364	4.79	0.27	3.54	0.24	2.45	0.48	26.86	0.11	9.20	0.24	16.73	0.45
365	4.53	0.26	3.78	0.24	2.35	0.50	26.75	0.11	9.44	0.24	17.18	0.45
366	4.27	0.26	4.03	0.25	2.26	0.53	26.62	0.13	9.69	0.25	17.64	0.46
367	4.03	0.24	4.28	0.25	2.16	0.57	26.47	0.15	9.94	0.25	18.11	0.47
368	3.79	0.24	4.53	0.25	2.07	0.60	26.31	0.16	10.19	0.25	18.58	0.47
369	3.56	0.23	4.79	0.26	1.98	0.65	26.13	0.18	10.45	0.26	19.05	0.47
370	3.35	0.21	5.05	0.26	1.89	0.69	25.94	0.19	10.70	0.25	19.52	0.47
371	3.14	0.21	5.31	0.26	1.81	0.74	25.74	0.20	10.96	0.26	20.00	0.48
372	2.94	0.20	5.58	0.27	1.72	0.79	25.51	0.23	11.22	0.26	20.48	0.48
373	2.75	0.19	5.85	0.27	1.65	0.84	25.27	0.24	11.48	0.26	20.97	0.49
374	2.58	0.17	6.12	0.27	1.57	0.90	25.02	0.25	11.75	0.27	21.46	0.49
375	2.41	0.17	6.39	0.27	1.50	0.96	24.75	0.27	12.01	0.26	21.95	0.49
376	2.24	0.17	6.66	0.27	1.42	1.02	24.46	0.29	12.28	0.27	22.44	0.49
377	2.09	0.15	6.94	0.28	1.36	1.08	24.16	0.30	12.55	0.27	22.94	0.50
378	1.95	0.14	7.22	0.28	1.29	1.15	23.84	0.32	12.82	0.27	23.44	0.50
379	1.81	0.14	7.49	0.27	1.23	1.21	23.51	0.33	13.09	0.27	23.95	0.51
380	1.68	0.13	7.77	0.28	1.17	1.28	23.16	0.35	13.36	0.27	24.46	0.51
381	1.56	0.12	8.04	0.27	1.11	1.35	22.80	0.36	13.63	0.27	24.97	0.51
382	1.45	0.11	8.32	0.28	1.06	1.42	22.42	0.38	13.91	0.28	25.48	0.51
383	1.34	0.11	8.60	0.28	1.01	1.49	22.03	0.39	14.19	0.28	25.99	0.51
384	1.24	0.10	8.88	0.28	0.96	1.57	21.62	0.41	14.46	0.27	26.51	0.52
385	1.15	0.09	9.15	0.27	0.92	1.64	21.21	0.41	14.74	0.28	27.03	0.52
386	1.06	0.09	9.43	0.28	0.87	1.72	20.77	0.44	15.02	0.28	27.56	0.53
387	0.98	0.08	9.70	0.27	0.83	1.79	20.33	0.44	15.30	0.28	28.08	0.52
388	0.91	0.07	9.98	0.28	0.79	1.87	19.88	0.45	15.58	0.28	28.61	0.53
389	0.84	0.07	10.25	0.27	0.76	1.95	19.42	0.46	15.86	0.28	29.13	0.52
390	0.78	0.06	10.52	0.27	0.72	2.03	18.94	0.48	16.15	0.29	29.66	0.53
391	0.72	0.06	10.79	0.27	0.70	2.10	18.46	0.48	16.43	0.28	30.19	0.53
392	0.67	0.05	11.06	0.27	0.67	2.18	17.97	0.49	16.71	0.28	30.72	0.53
393	0.62	0.05	11.34	0.28	0.64	2.26	17.47	0.50	17.00	0.29	31.25	0.53
394	0.57	0.05	11.60	0.26	0.62	2.35	16.97	0.50	17.28	0.28	31.79	0.54
395	0.53	0.04	11.88	0.28	0.60	2.43	16.45	0.52	17.57	0.29	32.32	0.53
396	0.50	0.03	12.14	0.26	0.59	2.51	15.94	0.51	17.86	0.29	32.86	0.54
397	0.46	0.04	12.42	0.28	0.57	2.59	15.42	0.52	18.14	0.28	33.39	0.53
398	0.43	0.03	12.68	0.26	0.56	2.67	14.90	0.52	18.43	0.29	33.93	0.54
399	0.41	0.02	12.95	0.27	0.55	2.75	14.37	0.53	18.71	0.28	34.46	0.53
400	0.38	0.03	13.22	0.27	0.54	2.83	13.85	0.52	19.00	0.29	35.00	0.54

Table.	VIII.	IX.	X.	XI.	XII.	XIII.	Table.	VIII.	IX.	X.	XI.	XII.	XIII.
Arg.	4	5	6	7	8	9	Arg.	4	5	6	7	8	9
	δv_4	δv_5	δv_6	δv_7	δv_8	δv_9		δv_4	δv_5	δv_6	δv_7	δv_8	δv_9
	"	"	"	"	"	"		"	"	"	"	"	"
0	0.08	0.15	0.94	0.94	0.30	0.80	200	1.12	0.05	0.06	1.26	0.10	0.20
10	0.08	0.14	0.96	1.06	0.28	0.84	210	1.12	0.06	0.04	1.14	0.12	0.16
20	0.10	0.13	0.97	1.19	0.26	0.88	220	1.10	0.07	0.03	1.01	0.14	0.12
30	0.13	0.12	0.96	1.31	0.24	0.91	230	1.07	0.08	0.04	0.89	0.16	0.09
40	0.18	0.10	0.95	1.43	0.22	0.93	240	1.02	0.10	0.05	0.77	0.18	0.07
50	0.23	0.09	0.93	1.54	0.20	0.94	250	0.97	0.11	0.07	0.66	0.20	0.06
60	0.29	0.08	0.89	1.64	0.18	0.94	260	0.91	0.12	0.11	0.56	0.22	0.06
70	0.36	0.07	0.85	1.72	0.15	0.93	270	0.84	0.13	0.15	0.48	0.25	0.07
80	0.44	0.06	0.79	1.79	0.13	0.90	280	0.76	0.14	0.21	0.41	0.27	0.10
90	0.52	0.05	0.73	1.85	0.12	0.87	290	0.68	0.15	0.27	0.35	0.28	0.13
100	0.60	0.04	0.67	1.88	0.10	0.83	300	0.60	0.16	0.33	0.32	0.30	0.17
110	0.68	0.04	0.60	1.90	0.09	0.78	310	0.52	0.16	0.40	0.30	0.31	0.22
120	0.76	0.03	0.52	1.90	0.08	0.72	320	0.44	0.17	0.48	0.30	0.32	0.28
130	0.84	0.03	0.45	1.87	0.07	0.66	330	0.36	0.17	0.55	0.33	0.33	0.34
140	0.91	0.03	0.38	1.83	0.06	0.59	340	0.29	0.17	0.62	0.37	0.34	0.41
150	0.97	0.03	0.31	1.77	0.06	0.52	350	0.23	0.17	0.69	0.43	0.34	0.48
160	1.02	0.03	0.25	1.69	0.06	0.45	360	0.18	0.17	0.75	0.51	0.34	0.55
170	1.07	0.03	0.19	1.60	0.07	0.38	370	0.13	0.17	0.81	0.60	0.33	0.62
180	1.10	0.04	0.14	1.50	0.07	0.32	380	0.10	0.16	0.86	0.70	0.33	0.68
190	1.12	0.05	0.09	1.40	0.09	0.26	390	0.08	0.15	0.91	0.80	0.31	0.74
200	1.12	0.05	0.06	1.26	0.10	0.20	400	0.08	0.15	0.94	0.94	0.30	0.80

TABLE XIV.

If the date is earlier than 1779, Jan. 4, or later than 1943, Oct. 15, the values of $P_{s.1}$ and $P_{c.1}$ must be corrected as follows, the argument being the year:

Year.	$\Delta P_{s.1}$	$\Delta P_{c.1}$	Year.	$\Delta P_{s.1}$	$\Delta P_{c.1}$	Year.	$\Delta P_{s.1}$	$\Delta P_{c.1}$
							"	"
1614.2	— 56.83	— 31.88	1700.0	— 65.57	— 24.94	1943.8	+ 72.87	+ 16.62
1620.0	— 57.44	— 31.46	1710.0	— 66.53	— 24.02	1950.0	+ 73.39	+ 15.88
1630.0	— 58.49	— 30.73	1720.0	— 67.50	— 23.04	1960.0	+ 74.21	+ 14.68
1640.0	— 59.52	— 29.98	1730.0	— 68.46	— 22.04	1970.0	+ 75.02	+ 13.44
1650.0	— 60.56	— 29.18	1740.0	— 69.40	— 21.00	1980.0	+ 75.81	+ 12.18
1660.0	— 61.58	— 28.36	1750.0	— 70.32	— 19.92	1990.0	+ 76.58	+ 10.88
1670.0	— 62.60	— 27.54	1760.0	— 71.22	— 18.82	2000.0	+ 77.32	+ 9.52
1680.0	— 63.62	— 26.70	1770.0	— 72.10	— 17.68			
1690.0	— 64.60	— 25.84	1779.0	— 72.87	— 16.62			

Between 1779 and 1943, P_s and P_c require no correction. For dates earlier than 1614 or later than 2000, the corrections must be computed from the formulæ.

TABLE XV.

EQUATION OF THE CENTRE.

l	Equation.	Diff.	l	Equation.	Diff.	l	Equation.	Diff.
°	° ′ ″	″	°	° ′ ″	″	°	° ′ ″	″
180	1 38 54.42		225	0 57 42.12		270	0 17 6.17	
181	1 38 10.17	44.25	226	0 56 41.74	60.38	271	0 16 24.17	42.00
182	1 37 25.21	44.96	227	0 55 41.40	60.34	272	0 15 42.92	41.25
183	1 36 39.56	45.65	228	0 54 41.12	60.28	273	0 15 2.45	40.47
184	1 35 53.23	46.33	229	0 53 40.90	60.22	274	0 14 22.76	39.69
185	1 35 6.24	46.99	230	0 52 40.78	60.12	275	0 13 43.86	38.90
186	1 34 18.60	47.64	231	0 51 40.76	60.02	276	0 13 5.77	38.09
187	1 33 30.33	48.27	232	0 50 40.87	59.89	277	0 12 28.50	37.27
188	1 32 41.43	48.90	233	0 49 41.12	59.75	278	0 11 52.06	36.44
189	1 31 51.93	49.50	234	0 48 41.53	59.59	279	0 11 16.46	35.60
190	1 31 1.84	50.09	235	0 47 42.12	59.41	280	0 10 41.71	34.75
191	1 30 11.17	50.67	236	0 46 42.90	59.22	281	0 10 7.82	33.89
192	1 29 19.93	51.24	237	0 45 43.90	59.00	282	0 9 34.81	33.01
193	1 28 28.15	51.78	238	0 44 45.13	58.77	283	0 9 2.69	32.12
194	1 27 35.84	52.31	239	0 43 46.61	58.52	284	0 8 31.46	31.23
195	1 26 43.02	52.82	240	0 42 48.36	58.25	285	0 8 1.13	30.33
196	1 25 49.70	53.32	241	0 41 50.39	57.97	286	0 7 31.72	29.41
197	1 24 55.90	53.80	242	0 40 52.72	57.67	287	0 7 3.23	28.49
198	1 24 1.64	54.26	243	0 39 55.36	57.36	288	0 6 35.67	27.56
199	1 23 6.93	54.71	244	0 38 58.34	57.02	289	0 6 9.06	26.61
200	1 22 11.78	55.15	245	0 38 1.67	56.67	290	0 5 43.39	25.67
201	1 21 16.21	55.57	246	0 37 5.37	56.30	291	0 5 18.68	24.71
202	1 20 20.25	55.96	247	0 36 9.45	55.92	292	0 4 54.94	23.74
203	1 19 23.90	56.35	248	0 35 13.94	55.51	293	0 4 32.18	22.76
204	1 18 27.19	56.71	249	0 34 18.84	55.10	294	0 4 10.40	21.78
205	1 17 30.13	57.06	250	0 33 24.18	54.66	295	0 3 49.60	20.80
206	1 16 32.73	57.40	251	0 32 29.97	54.21	296	0 3 29.80	19.80
207	1 15 35.02	57.71	252	0 31 36.22	53.75	297	0 3 11.00	18.80
208	1 14 37.01	58.01	253	0 30 42.96	53.26	298	0 2 53.21	17.79
209	1 13 38.72	58.29	254	0 29 50.19	52.77	299	0 2 36.44	16.77
210	1 12 40.16	58.56	255	0 28 57.94	52.25	300	0 2 20.68	15.76
211	1 11 41.35	58.81	256	0 28 6.22	51.72	301	0 2 5.95	14.73
212	1 10 42.32	59.03	257	0 27 15.05	51.17	302	0 1 52.26	13.69
213	1 9 43.08	59.24	258	0 26 24.43	50.62	303	0 1 39.60	12.66
214	1 8 43.64	59.44	259	0 25 34.40	50.03	304	0 1 27.97	11.63
215	1 7 44.03	59.61	260	0 24 44.95	49.45	305	0 1 17.39	10.58
216	1 6 44.26	59.77	261	0 23 56.11	48.84	306	0 1 7.85	9.54
217	1 5 44.35	59.91	262	0 23 7.89	48.22	307	0 0 59.36	8.49
218	1 4 44.32	60.03	263	0 22 20.31	47.58	308	0 0 51.93	7.43
219	1 3 44.18	60.14	264	0 21 33.37	46.94	309	0 0 45.55	6.38
220	1 2 43.95	60.23	265	0 20 47.10	46.27	310	0 0 40.24	5.31
221	1 1 43.65	60.30	266	0 20 1.50	45.60	311	0 0 36.00	4.24
222	1 0 43.30	60.35	267	0 19 16.60	44.90	312	0 0 32.82	3.18
223	0 59 42.92	60.38	268	0 18 32.40	44.20	313	0 0 30.71	2.11
224	0 58 42.52	60.40	269	0 17 48.92	43.48	314	0 0 29.67	1.04
225	0 57 42.12	60.40	270	0 17 6.17	42.75	315	0 0 29.70	0.03

TABLE XV.

Equation of the Centre (Continued).

l	Equation.	Diff.	*l*	Equation.	Diff.	*l*	Equation.	Diff.
°	° ′ ″	″	°	° ′ ″	″	°	° ′ ″	″
315	0 0 29.70		0	0 18 11.18		45	1 0 24.20	
316	0 0 30.80	1.10	1	0 18 55.91	44.73	46	1 1 26.40	62.20
317	0 0 32.96	2.16	2	0 19 41.40	45.49	47	1 2 28.56	62.16
318	0 0 36.20	3.24	3	0 20 27.63	46.23	48	1 3 30.65	62.09
319	0 0 40.50	4.30	4	0 21 14.59	46.96	49	1 4 32.67	62.02
320	0 0 45.88	5.38	5	0 22 2.26	47.67	50	1 5 34.58	61.91
321	0 0 52.33	6.45	6	0 22 50.63	48.37	51	1 6 36.37	61.79
322	0 0 59.84	7.51	7	0 23 39.68	49.05	52	1 7 38.02	61.65
323	0 1 8.42	8.58	8	0 24 29.40	49.72	53	1 8 39.51	61.49
324	0 1 18.06	9.64	9	0 25 19.78	50.38	54	1 9 40.82	61.31
325	0 1 28.76	10.70	10	0 26 10.79	51.01	55	1 10 41.93	61.11
326	0 1 40.52	11.76	11	0 27 2.43	51.64	56	1 11 42.82	60.89
327	0 1 53.34	12.82	12	0 27 54.67	52.24	57	1 12 43.46	60.64
328	0 2 7.21	13.87	13	0 28 47.50	52.83	58	1 13 43.85	60.39
329	0 2 22.13	14.92	14	0 29 40.91	53.41	59	1 14 43.96	60.11
330	0 2 38.10	15.97	15	0 30 34.87	53.96	60	1 15 43.78	59.82
331	0 2 55.11	17.01	16	0 31 29.38	54.51	61	1 16 43.28	59.50
332	0 3 13.16	18.05	17	0 32 24.41	55.03	62	1 17 42.45	59.17
333	0 3 32.24	19.08	18	0 33 19.95	55.54	63	1 18 41.27	58.82
334	0 3 52.35	20.11	19	0 34 15.97	56.02	64	1 19 39.71	58.44
335	0 4 13.48	21.13	20	0 35 12.47	56.50	65	1 20 37.76	58.05
336	0 4 35.62	22.14	21	0 36 9.42	56.95	66	1 21 35.41	57.65
337	0 4 58.78	23.16	22	0 37 6.81	57.39	67	1 22 32.62	57.21
338	0 5 22.94	24.16	23	0 38 4.61	57.80	68	1 23 29.39	56.77
339	0 5 48.09	25.15	24	0 39 2.81	58.20	69	1 24 25.70	56.31
340	0 6 14.24	26.15	25	0 40 1.40	58.59	70	1 25 21.53	55.83
341	0 6 41.37	27.13	26	0 41 0.35	58.95	71	1 26 16.87	55.34
342	0 7 9.47	28.10	27	0 41 59.64	59.29	72	1 27 11.69	54.82
343	0 7 38.54	29.07	28	0 42 59.26	59.62	73	1 28 5.97	54.28
344	0 8 8.57	30.03	29	0 43 59.19	59.93	74	1 28 59.71	53.74
345	0 8 39.54	30.97	30	0 44 59.41	60.22	75	1 29 52.88	53.17
346	0 9 11.46	31.92	31	0 45 59.90	60.49	76	1 30 45.46	52.58
347	0 9 44.30	32.84	32	0 47 0.63	60.73	77	1 31 37.45	51.99
348	0 10 18.07	33.77	33	0 48 1.60	60.97	78	1 32 28.82	51.37
349	0 10 52.74	34.67	34	0 49 2.78	61.18	79	1 33 19.55	50.73
350	0 11 28.32	35.58	35	0 50 4.16	61.38	80	1 34 9.64	50.09
351	0 12 4.79	36.47	36	0 51 5.71	61.55	81	1 34 59.06	49.42
352	0 12 42.14	37.35	37	0 52 7.41	61.70	82	1 35 47.80	48.74
353	0 13 20.35	38.21	38	0 53 9.25	61.84	83	1 36 35.85	48.05
354	0 13 59.42	39.07	39	0 54 11.20	61.95	84	1 37 23.19	47.34
355	0 14 39.34	39.92	40	0 55 13.25	62.05	85	1 38 9.80	46.61
356	0 15 20.09	40.75	41	0 56 15.37	62.12	86	1 38 55.67	45.87
357	0 16 1.66	41.57	42	0 57 17.54	62.17	87	1 39 40.80	45.13
358	0 16 44.04	42.38	43	0 58 19.75	62.21	88	1 40 25.15	44.35
359	0 17 27.22	43.18	44	0 59 21.98	62.23	89	1 41 8.73	43.58
360	0 18 11.18	43.96	45	1 0 24.20	62.22	90	1 41 51.51	42.78

TABLE XV.

EQUATION OF THE CENTRE (Concluded).

l	Equation.	Diff.	l	Equation.	Diff.	l	Equation.	Diff.
°	° ′ ″	″	°	° ′ ″	″	°	° ′ ″	″
90	1 41 51.51		120	1 56 10.41		150	1 54 54.44	
91	1 42 33.48	41.97	121	1 56 23.47	13.06	151	1 54 35.48	18.96
92	1 43 14.63	41.15	122	1 56 35.47	12.00	152	1 54 15.50	19.98
93	1 43 54.95	40.32	123	1 56 46.39	10.92	153	1 53 54.51	20.99
94	1 44 34.42	39.47	124	1 56 56.24	9.85	154	1 53 32.52	21.99
95	1 45 13.04	38.62	125	1 57 5.02	8.78	155	1 53 9.53	22.99
96	1 45 50.78	37.74	126	1 57 12.72	7.70	156	1 52 45.55	23.98
97	1 46 27.65	36.87	127	1 57 19.34	6.62	157	1 52 20.59	24.96
98	1 47 3.62	35.97	128	1 57 24.87	5.53	158	1 51 54.66	25.93
99	1 47 38.69	35.07	129	1 57 29.33	4.46	159	1 51 27.76	26.90
100	1 48 12.85	34.16	130	1 57 32.70	3.37	160	1 50 59.91	27.85
101	1 48 46.08	33.23	131	1 57 34.99	2.29	161	1 50 31.11	28.80
102	1 49 18.38	32.30	132	1 57 36.20	1.21	162	1 50 1.38	29.73
103	1 49 49.73	31.35	133	1 57 36.32	0.12	163	1 49 30.71	30.67
104	1 50 20.13	30.40	134	1 57 35.36	0.96	164	1 48 59.13	31.58
105	1 50 49.57	29.44	135	1 57 33.31	2.05	165	1 48 26.64	32.49
106	1 51 18.03	28.46	136	1 57 30.19	3.12	166	1 47 53.25	33.39
107	1 51 45.52	27.49	137	1 57 25.98	4.21	167	1 47 18.97	34.28
108	1 52 12.01	26.49	138	1 57 20.70	5.28	168	1 46 43.82	35.15
109	1 52 37.51	25.50	139	1 57 14.34	6.36	169	1 46 7.80	36.02
110	1 53 2.01	24.50	140	1 57 6.91	7.43	170	1 45 30.92	36.88
111	1 53 25.50	23.49	141	1 56 58.41	8.50	171	1 44 53.20	37.72
112	1 53 47.97	22.47	142	1 56 48.84	9.57	172	1 44 14.65	38.55
113	1 54 9.42	21.45	143	1 56 38.21	10.63	173	1 43 35.28	39.37
114	1 54 29.84	20.42	144	1 56 26.52	11.69	174	1 42 55.10	40.18
115	1 54 49.22	19.38	145	1 56 13.78	12.74	175	1 42 14.13	40.97
116	1 55 7.56	18.34	146	1 55 59.99	13.79	176	1 41 32.38	41.75
117	1 55 24.85	17.29	147	1 55 45.16	14.83	177	1 40 49.85	42.53
118	1 55 41.10	16.25	148	1 55 29.28	15.88	178	1 40 6.57	43.28
119	1 55 56.28	15.18	149	1 55 12.38	16.90	179	1 39 22.55	44.02
120	1 56 10.41	14.13	150	1 54 54.44	17.94	180	1 38 37.80	44.75

TABLE XVI.

Reduction to the Ecliptic.

Argument *u*.				1800	1900	2000	Diff. 100Y.	Argument *u*.				1800	1900	2000	Diff. 100Y.
°	°	°	°	"	"	"	"	°	°	°	°	"	"	"	"
0	90	180	270	60.00	60.00	60.00	0.00	135	315	135	315	110.23	109.72	109.21	0.51
1	89	181	269	58.25	58.26	58.28	0.01	134	316	136	314	110.20	109.70	109.18	0.51
2	88	182	268	56.50	56.53	56.57	0.03	133	317	137	313	110.11	109.61	109.10	0.51
3	87	183	267	54.75	54.80	54.84	0.05	132	318	138	312	109.96	109.45	108.95	0.50
4	86	184	266	53.01	53.07	53.14	0.07	131	319	139	311	109.74	109.24	108.74	0.50
5	85	185	265	51.27	51.36	51.45	0.09	130	320	140	310	109.47	108.97	108.47	0.50
6	84	186	264	49.55	49.66	49.76	0.11	129	321	141	309	109.14	108.64	108.14	0.50
7	83	187	263	47.85	47.97	48.09	0.12	128	322	142	308	108.74	108.25	107.76	0.49
8	82	188	262	46.15	46.29	46.43	0.14	127	323	143	307	108.29	107.80	107.32	0.49
9	81	189	261	44.48	44.63	44.79	0.15	126	324	144	306	107.78	107.30	106.81	0.48
10	80	190	260	42.82	42.99	43.16	0.17	125	325	145	305	107.21	106.73	106.25	0.48
11	79	191	259	41.18	41.37	41.56	0.19	124	326	146	304	106.58	106.11	105.64	0.47
12	78	192	258	39.57	39.78	39.98	0.21	123	327	147	303	105.90	105.44	104.97	0.46
13	77	193	257	37.98	38.20	38.43	0.22	122	328	148	302	105.15	104.70	104.24	0.45
14	76	194	256	36.42	36.66	36.90	0.24	121	329	149	301	104.35	103.91	103.46	0.44
15	75	195	255	34.88	35.14	35.40	0.26	120	330	150	300	103.50	103.06	102.62	0.44
16	74	196	254	33.38	33.65	33.93	0.27	119	331	151	299	102.60	102.17	101.74	0.43
17	73	197	253	31.91	32.20	32.49	0.29	118	332	152	298	101.64	101.22	100.80	0.42
18	72	198	252	30.47	30.78	31.08	0.31	117	333	153	297	100.64	100.23	99.82	0.41
19	71	199	251	29.07	29.39	29.71	0.32	116	334	154	296	99.58	99.18	98.78	0.40
20	70	200	250	27.71	28.03	28.37	0.33	115	335	155	295	98.48	98.09	97.69	0.39
21	69	201	249	26.39	26.73	27.07	0.34	114	336	156	294	97.33	96.95	96.57	0.38
22	68	202	248	25.11	25.46	25.82	0.35	113	337	157	293	96.13	95.77	95.40	0.36
23	67	203	247	23.87	24.23	24.60	0.36	112	338	158	292	94.89	94.54	94.18	0.35
24	66	204	246	22.67	23.05	23.43	0.38	111	339	159	291	93.61	93.27	92.93	0.34
25	65	205	245	21.52	21.91	22.31	0.39	110	340	160	290	92.29	91.96	91.63	0.33
26	64	206	244	20.42	20.82	21.22	0.40	109	341	161	289	90.93	90.61	90.29	0.32
27	63	207	243	19.36	19.77	20.18	0.41	108	342	162	288	89.53	89.22	88.92	0.31
28	62	208	242	18.36	18.78	19.20	0.42	107	343	163	287	88.09	87.80	87.51	0.29
29	61	209	241	17.40	17.83	18.26	0.43	106	344	164	286	86.62	86.35	86.07	0.27
30	60	210	240	16.50	16.94	17.38	0.44	105	345	165	285	85.12	84.86	84.60	0.26
31	59	211	239	15.65	16.09	16.54	0.44	104	346	166	284	83.58	83.34	83.10	0.24
32	58	212	238	14.85	15.30	15.76	0.45	103	347	167	283	82.02	81.80	81.57	0.22
33	57	213	237	14.10	14.56	15.03	0.46	102	348	168	282	80.43	80.22	80.02	0.21
34	56	214	236	13.42	13.89	14.36	0.47	101	349	169	281	78.82	78.63	78.44	0.19
35	55	215	235	12.79	13.27	13.75	0.48	100	350	170	280	77.18	77.01	76.84	0.17
36	54	216	234	12.22	12.70	13.19	0.48	99	351	171	279	75.52	75.37	75.21	0.15
37	53	217	233	11.71	12.20	12.68	0.49	98	352	172	278	73.85	73.71	73.57	0.14
38	52	218	232	11.26	11.75	12.24	0.49	97	353	173	277	72.15	72.03	71.91	0.12
39	51	219	231	10.86	11.36	11.86	0.50	96	354	174	276	70.45	70.34	70.24	0.10
40	50	220	230	10.53	11.03	11.53	0.50	95	355	175	275	68.73	68.64	68.55	0.09
41	49	221	229	10.26	10.76	11.26	0.50	94	356	176	274	66.99	66.93	66.86	0.07
42	48	222	228	10.04	10.55	11.05	0.50	93	357	177	273	65.25	65.20	65.16	0.05
43	47	223	227	9.89	10.39	10.90	0.51	92	358	178	272	63.50	63.47	63.43	0.03
44	46	224	226	9.80	10.30	10.82	0.51	91	359	179	271	61.75	61.74	61.72	0.01
45	45	225	225	9.77	10.28	10.79	0.51	90	0	180	270	60.00	60.00	60.00	0.00

TABLE XVII.

Coefficients for Perturbations of Log. Radius Vector.

Argument 1.

	0		50		100		150		200		250		300		350	
	$R_{s.1}$	$R_{c.1}$	$R_{s.1}$	$R_{c.1}$	$R_{s.1}$	$R_{c.1}$	$R_{s.1}$	$R_{c.1}$	$R_{s.1}$	$R_{c.1}$	$R_{s.1}$	$R_{c.1}$	$R_{s.1}$	$R_{c.1}$	$R_{s.1}$	$R_{c.1}$
0	131	23	40	26	34	170	176	142	131	45	78	142	212	170	214	26
1	130	23	37	28	36	172	178	139	129	45	79	145	214	168	212	24
2	130	23	35	30	39	174	180	136	126	45	81	147	216	166	209	22
3	129	23	32	32	41	176	181	133	124	45	82	150	218	163	207	20
4	129	23	30	35	44	178	183	130	122	46	84	152	220	160	204	18
5	128	23	28	37	46	180	184	127	120	46	86	155	222	158	202	17
6	127	23	26	39	49	181	185	124	118	47	88	157	224	156	199	16
7	127	22	24	42	51	183	186	121	116	47	91	160	226	153	197	15
8	126	21	22	45	54	185	186	119	113	48	93	162	227	151	194	15
9	126	21	20	48	57	185	187	116	111	49	96	165	229	148	191	14
10	125	20	18	51	60	186	187	113	109	50	98	167	230	146	189	13
11	124	19	16	54	63	186	187	110	107	51	100	169	232	143	187	12
12	123	18	14	57	66	187	187	108	104	52	103	171	233	140	184	11
13	123	18	13	60	70	187	187	105	102	53	105	173	235	138	182	10
14	122	17	11	64	73	188	187	102	100	54	108	175	237	135	179	9
15	121	16	10	67	76	188	187	99	97	56	110	177	238	132	177	9
16	120	15	9	70	79	189	187	96	95	57	113	178	239	129	175	9
17	118	15	8	73	82	190	186	93	93	59	116	180	240	126	172	9
18	117	14	8	76	86	191	185	90	92	60	119	181	241	122	170	9
19	115	13	7	79	89	192	184	88	90	61	122	183	241	119	167	9
20	113	13	7	82	92	192	183	85	89	63	125	184	242	116	165	9
21	111	13	6	85	95	192	183	83	87	65	128	185	242	113	163	9
22	110	13	6	88	98	190	182	80	85	67	131	186	243	109	161	10
23	108	12	5	92	102	189	182	78	83	70	134	187	244	105	160	10
24	106	12	5	95	105	189	181	76	81	72	137	187	244	102	158	11
25	104	11	4	98	108	188	180	74	80	74	140	188	244	98	156	11
26	102	11	4	102	111	187	178	72	79	76	143	189	244	95	154	12
27	100	10	4	105	115	187	177	70	78	78	146	190	244	92	152	12
28	98	10	5	109	118	186	175	67	77	80	150	191	244	88	150	12
29	95	9	5	113	121	185	173	65	76	83	153	192	243	85	148	13
30	93	9	6	116	125	184	171	63	75	85	156	192	243	82	147	13
31	91	9	6	119	128	183	170	61	74	88	158	192	243	79	146	14
32	88	9	7	122	131	181	168	60	73	90	161	191	242	76	144	14
33	86	9	7	126	134	180	167	59	72	93	164	190	242	73	143	15
34	83	9	8	129	137	178	166	57	72	96	167	189	241	70	142	15
35	81	9	8	132	140	177	165	56	71	99	170	188	240	67	141	16
36	78	9	9	135	143	175	163	54	70	102	173	188	239	64	140	17
37	76	10	10	138	146	173	160	53	70	105	176	187	238	60	139	18
38	73	11	12	140	149	171	158	51	69	108	180	187	237	57	138	18
39	70	12	13	143	151	169	155	50	69	110	183	186	235	54	138	19
40	67	13	16	146	154	167	153	49	69	113	186	186	234	51	137	20
41	64	14	17	148	156	165	151	48	69	116	189	185	232	48	136	21
42	62	15	19	151	159	162	148	48	70	119	191	184	230	45	136	21
43	59	15	21	153	162	160	146	48	70	121	194	183	228	42	135	22
44	57	16	22	156	164	157	144	47	71	124	197	181	226	39	134	22
45	54	17	24	158	166	155	142	47	72	127	200	180	224	37	134	22
46	51	18	26	160	168	152	140	46	73	130	202	178	222	35	133	23
47	49	20	28	163	170	150	138	45	74	133	205	176	220	32	133	23
48	46	22	30	166	172	147	135	45	75	136	207	174	218	30	132	23
49	43	24	32	168	174	145	133	45	77	139	210	172	216	28	131	23
50	40	26	34	170	176	142	131	45	78	142	212	170	214	26	131	23

Note.—Before 1779 and after 1943, we have

$\triangle R_{s.1} = \quad 10.53 \triangle P_{c.1}$ 1614 — 1778 $\delta \log r = - 314.$

$\triangle R_{c.1} = - 10.53 \triangle P_{s.1}$ 1943 — 2108 $\delta \log r = + 314.$

Perturbations of Logarithm of Radius Vector.

	TABLE XVIII.				TABLE XIX.				TABLE XX.				
	Argument 1.				Argument 2.				Argument 3.				
Arg.	0	50	100	150	0	50	100	150	0	50	100	150	
0	743	387	58	5	801	681	396	119	1401	1196	700	204	50
1	743	378	54	5	801	676	390	115	1401	1188	689	196	49
2	743	369	51	6	801	671	383	111	1400	1180	678	188	48
3	742	361	48	6	801	666	377	107	1400	1172	667	181	47
4	740	352	44	7	800	662	371	103	1399	1163	656	174	46
5	738	343	41	7	800	657	365	99	1398	1155	645	167	45
6	736	334	38	7	800	652	359	95	1397	1147	634	160	44
7	733	325	35	8	799	647	353	91	1396	1138	623	153	43
8	730	316	33	9	798	642	346	88	1395	1130	613	146	42
9	726	307	30	9	797	637	340	84	1393	1121	602	139	41
10	722	298	28	10	796	632	334	80	1392	1112	591	133	40
11	718	290	26	11	795	627	328	77	1390	1103	580	127	39
12	713	282	24	11	794	621	322	73	1388	1094	570	121	38
13	708	274	23	12	793	616	316	70	1386	1085	559	115	37
14	702	267	21	13	791	610	310	67	1384	1075	548	109	36
15	696	259	19	14	790	605	304	64	1382	1066	537	103	35
16	690	251	17	14	788	600	298	61	1379	1057	526	97	34
17	684	244	16	15	787	594	292	57	1377	1047	515	91	33
18	678	236	14	15	785	589	286	54	1374	1038	504	85	32
19	672	229	13	16	783	583	280	51	1370	1028	493	80	31
20	665	222	12	16	781	578	274	48	1367	1018	483	75	30
21	658	215	10	16	779	573	268	45	1364	1008	473	70	29
22	651	207	9	17	777	567	262	43	1360	998	463	65	28
23	643	200	8	18	775	561	257	40	1356	988	452	60	27
24	635	192	6	19	772	555	251	38	1352	978	442	56	26
25	626	185	5	20	770	549	245	36	1348	968	432	52	25
26	617	179	4	20	767	543	239	34	1344	958	422	48	24
27	608	172	3	20	765	537	234	31	1339	948	412	44	23
28	599	166	3	21	762	532	228	29	1335	937	402	40	22
29	590	160	2	21	759	526	223	27	1330	927	392	36	21
30	580	154	2	21	756	520	218	25	1325	917	382	33	20
31	570	148	2	22	753	513	213	23	1320	907	372	30	19
32	561	143	2	22	750	507	207	22	1315	896	362	26	18
33	551	137	2	23	747	500	202	20	1309	885	353	23	17
34	542	131	3	23	743	494	196	19	1303	874	343	21	16
35	532	125	3	24	740	488	191	18	1297	863	334	18	15
36	522	120	3	24	736	482	186	17	1291	852	325	16	14
37	513	115	3	25	733	476	181	16	1285	841	315	14	13
38	503	110	4	25	729	470	176	14	1279	830	306	12	12
39	494	105	4	26	726	464	171	13	1273	820	297	10	11
40	484	100	4	26	722	458	166	12	1267	809	288	8	10
41	474	95	4	26	718	452	161	11	1261	798	279	7	9
42	464	91	4	27	714	446	156	10	1254	787	270	5	8
43	455	86	4	27	711	439	151	10	1247	777	262	4	7
44	445	81	4	27	707	432	146	9	1240	766	253	3	6
45	435	77	4	28	703	427	141	8	1233	755	245	2	5
46	425	73	4	28	699	421	137	8	1226	744	237	1	4
47	416	69	4	28	694	415	132	8	1219	733	228	0	3
48	406	65	5	29	690	408	128	7	1212	722	220	0	2
49	397	61	5	29	685	402	123	7	1204	711	212	0	1
50	387	58	5	29	681	396	119	7	1196	700	204	0	0
	350	300	250	200	350	300	250	200	350	300	250	200	Arg.

TABLE XXI.

Principal Term of the Logarithm of the Radius Vector.

Argument l.

l	1.4		l	1.4		l	1.4	
°			°			°		
180	806676		240	815373		300	788978	
181	7112	436	241	5192	181	301	8352	626
182	7540	428	242	5001	191	302	7722	630
183	7960	420	243	4799	202	303	7089	633
184	8371	411	244	4587	212	304	6455	634
185	808775	404	245	814364	223	305	785818	637
186	9169	394	246	4130	234	306	5179	639
187	9554	385	247	3885	245	307	4538	641
188	9931	377	248	3631	254	308	3896	642
189	810299	368	249	3366	265	309	3252	644
190	810657	358	250	813092	274	310	782607	645
191	1005	348	251	2807	285	311	1961	646
192	1344	339	252	2513	294	312	1314	647
193	1674	330	253	2208	305	313	0667	647
194	1994	320	254	1894	314	314	0020	647
195	812305	311	255	811570	324	315	779373	647
196	2606	301	256	1237	333	316	8726	647
197	2897	291	257	0895	342	317	8079	647
198	3178	281	258	0543	352	318	7432	647
199	3449	271	259	0182	361	319	6786	646
200	813710	261	260	809812	370	320	776140	646
201	3961	251	261	9433	379	321	5495	645
202	4202	241	262	9045	388	322	4851	644
203	4432	230	263	8648	397	323	4208	643
204	4653	221	264	8242	406	324	3567	641
205	814864	210	265	807827	415	325	772927	640
206	5064	200	266	7404	423	326	2289	638
207	5252	188	267	6972	432	327	1654	635
208	5430	178	268	6532	440	328	1022	632
209	5597	167	269	6084	448	329	0393	629
210	815753	156	270	805629	455	330	769767	626
211	5899	146	271	5166	463	331	9143	624
212	6034	135	272	4696	470	332	8522	621
213	6159	125	273	4217	479	333	7904	618
214	6273	114	274	3731	486	334	7290	614
215	816376	103	275	803236	495	335	766679	611
216	6468	92	276	2735	501	336	6072	607
217	6549	81	277	2227	508	337	5470	602
218	6619	70	278	1712	515	338	4872	598
219	6678	59	279	1191	521	339	4279	593
220	816727	49	280	800663	528	340	763691	588
221	6763	36	281	0128	535	341	3108	583
222	6789	26	282	799588	540	342	2530	578
223	6804	15	283	9041	547	343	1957	573
224	6807	3	284	8488	553	344	1389	568
225	816800	7	285	797930	558	345	760826	563
226	6782	18	286	7366	564	346	0270	556
227	6752	30	287	6796	570	347	759719	551
228	6711	41	288	6221	575	348	9174	545
229	6659	52	289	5641	580	349	8636	538
230	816596	63	290	795056	585	350	758104	532
231	6523	73	291	4467	589	351	7579	525
232	6439	84	292	3873	594	352	7061	518
233	6344	95	293	3274	599	353	6549	512
234	6238	106	294	2671	603	354	6045	504
235	816121	117	295	792063	608	355	755548	497
236	5993	128	296	1452	611	356	5059	489
237	5854	139	297	0838	614	357	4578	481
238	5704	150	298	0221	617	358	4104	474
239	5544	160	299	789601	620	359	3638	466
240	815373	171	300	788978	623	360	753181	457

TABLE XXI.

Principal Term of the Logarithm of the Radius Vector (Continued).

Argument l.

l	1.4		l	1.4		l	1.4	
°			°			°		
0	753181		60	744487		120	772170	
1	2731	450	61	4685	198	121	2811	641
2	2290	441	62	4894	209	122	3454	643
3	1858	432	63	5114	220	123	4098	644
4	1434	424	64	5346	232	124	4744	646
5	751020	415	65	745588	242	125	775392	648
6	0615	405	66	5841	253	126	6041	649
7	0218	397	67	6105	264	127	6691	650
8	749830	388	68	6380	275	128	7343	652
9	9452	378	69	6665	285	129	7996	653
10	749083	369	70	746961	296	130	778650	654
11	8724	359	71	7267	306	131	9303	653
12	8375	349	72	7584	317	132	9956	653
13	8036	339	73	7910	326	133	780609	653
14	7706	330	74	8246	336	134	1262	653
15	747386	320	75	748592	346	135	781915	653
16	7076	310	76	8948	356	136	2567	652
17	6777	299	77	9314	366	137	3219	652
18	6488	289	78	9689	375	138	3869	650
19	6210	278	79	750074	385	139	4518	649
20	745942	268	80	750469	395	140	785166	648
21	5685	257	81	0873	404	141	5812	646
22	5439	246	82	1286	413	142	6456	644
23	5204	235	83	1707	421	143	7098	642
24	4980	224	84	2137	430	144	7738	640
25	744766	214	85	752576	439	145	788376	638
26	4563	203	86	3024	448	146	9011	635
27	4372	191	87	3480	456	147	9642	631
28	4191	181	88	3944	464	148	790271	629
29	4021	170	89	4417	473	149	0897	626
30	743862	159	90	754897	480	150	791519	622
31	3715	147	91	5385	488	151	2138	619
32	3580	135	92	5881	496	152	2753	615
33	3456	124	93	6384	503	153	3364	611
34	3344	112	94	6895	511	154	3971	607
35	743242	102	95	757414	519	155	794575	604
36	3152	90	96	7940	526	156	5174	599
37	3073	79	97	8472	532	157	5768	594
38	3007	66	98	9010	538	158	6357	589
39	2953	54	99	9553	543	159	6941	584
40	742910	43	100	760103	550	160	797519	578
41	2879	31	101	0660	557	161	8093	574
42	2860	19	102	1223	563	162	8661	568
43	2852	8	103	1792	569	163	9223	562
44	2856	4	104	2367	575	164	799779	556
45	742872	16	105	762948	581	165	800330	551
46	2899	27	106	3533	585	166	0875	545
47	2937	38	107	4123	590	167	1413	538
48	2987	50	108	4718	595	168	1944	531
49	3048	61	109	5317	599	169	2468	524
50	743120	72	110	765922	605	170	802985	517
51	3204	84	111	6531	609	171	3496	511
52	3300	96	112	7143	612	172	4000	504
53	3408	108	113	7759	616	173	4497	497
54	3529	121	114	8379	620	174	4986	489
55	743661	132	115	769003	624	175	805468	482
56	3804	143	116	9631	628	176	5942	474
57	3958	154	117	770262	631	177	6408	466
58	4123	165	118	0895	633	178	6866	458
59	4300	177	119	1531	636	179	7317	451
60	744487	187	120	772170	639	180	807759	442

TABLE XXII.

Coefficients for Perturbations of Latitude.

Argument 1.

Arg.	0		100		200		300	
	$B_{s.1}$	$B_{c.1}$	$B_{s.1}$	$B_{c.1}$	$B_{s.1}$	$B_{c.1}$	$B_{s.1}$	$B_{c.1}$
	"	"	"	"	"	"	"	"
0	0.52	0.40	0.41	0.00	0.04	0.18	0.14	0.58
10	0.56	0.35	0.31	0.00	0.04	0.18	0.21	0.65
20	0.61	0.31	0.23	0.03	0.04	0.17	0.30	0.70
30	0.67	0.27	0.15	0.06	0.03	0.18	0.38	0.73
40	0.72	0.23	0.09	0.10	0.02	0.20	0.45	0.73
50	0.73	0.18	0.05	0.13	0.01	0.23	0.50	0.70
60	0.72	0.13	0.03	0.16	0.01	0.28	0.52	0.65
70	0.67	0.08	0.02	0.17	0.02	0.34	0.52	0.59
80	0.59	0.04	0.03	0.18	0.04	0.42	0.51	0.52
90	0.50	0.01	0.04	0.18	0.08	0.50	0.51	0.46
100	0.41	0.00	0.04	0.18	0.14	0.58	0.52	0.40

Perturbations of Latitude.

Arg.	TABLE XXIII.				TABLE XXIV.			
	Arg. 5.				Arg. 8.			
	0	100	200	300	0	100	200	300
	"	"	"	"	"	"	"	"
0	−0.30	+0.06	+0.30	−0.06	+0.04	+0.56	−0.04	−0.56
10	−0.29	+0.11	+0.29	−0.11	+0.13	+0.55	−0.13	−0.55
20	−0.27	+0.16	+0.27	−0.16	+0.21	+0.52	−0.21	−0.52
30	−0.24	+0.19	+0.24	−0.19	+0.29	+0.48	−0.29	−0.48
40	−0.21	+0.23	+0.21	−0.23	+0.36	+0.43	−0.36	−0.43
50	−0.17	+0.26	+0.17	−0.26	+0.43	+0.37	−0.43	−0.37
60	−0.12	+0.28	+0.12	−0.28	+0.48	+0.30	−0.48	−0.30
70	−0.08	+0.30	+0.08	−0.30	+0.52	+0.22	−0.52	−0.22
80	−0.03	+0.31	+0.03	−0.31	+0.55	+0.14	−0.55	−0.14
90	+0.02	+0.31	−0.02	−0.31	+0.56	+0.05	−0.56	−0.05
100	+0.06	+0.30	−0.06	−0.30	+0.56	−0.04	−0.56	+0.04

TABLE XXV.

Values of Sin i for every Ten Years.

Year.	1600		1700		1800		1900	
0	8.498705		8.496503		8.494292		8.492066	
10	8485	220	6282	221	4071	221	1842	224
20	8265	220	6061	221	3849	222	1619	223
30	8045	220	5840	221	3627	222	1395	224
40	7825	220	5619	221	3404	223	1171	224
50	8.497605	220	8.495398	221	8.493182	222	8.490947	224
60	7385	220	5177	221	2959	223	0723	224
70	7165	220	4956	221	2736	223	0498	225
80	6944	221	4735	221	2513	223	0274	224
90	6724	220	4513	222	2289	224	8.490049	225
100	8.496503	221	8.494292	221	8.492066	223	8.489824	225

PUBLISHED BY THE SMITHSONIAN INSTITUTION,

WASHINGTON, D. C.

JANUARY, 1866.